Reduction theorems for some global-local conjectures

Britta Späth

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Abstract. This survey covers various Clifford-theoretic aspects that give an understanding of the main common arguments behind the reduction theorems of some of the global-local counting conjectures in block theory. This applies in particular to the McKay conjecture, its blockwise version (Alperin-McKay) and the Alperin weight conjecture. This note grew out of a course given at the TU Kaiserslautern in the winter term 2013/2014 and a series of lectures given at the Centre Bernoulli (EPFL) in July 2016.

Introduction

In the representation theory of finite groups some of the oldest conjectures relate representation theoretic invariants of the group with those of their subgroups. In the last two decades a major step towards their verification was done by showing that these are consequences of conjectured stronger analogous statements for simple groups of Lie type. These so-called reduction theorems have been possible by a deeper understanding of Clifford theory and by clarifying how the representation theory of subgroups of simple groups is reflected by the representation theory of the whole group.

In the course of the years it crystallized that the language and notion of character triples and various relations between them are best adapted for the proof of those reduction theorems. We give here an introduction to this theory by providing a proof of the reduction theorem of the McKay conjecture in those terms, see Theorem 3.15 and its proof.

A crucial feature of character triples and the relations $\geq_c$, $\geq_b$ we define on them is that they apply to blocks. We give in particular the necessary material to prove a reduction theorem for the (blockwise) Alperin weight conjecture in those terms, see Theorem 4.22. We also show how the verification can be performed in a special case (simple groups of Lie type with regard to the defining characteristic).

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Notation

We use freely the usual notation from group theory. Our groups are finite. We write $H \leq G$, $H \lhd G$ to mean that H is a subgroup of G, respectively a normal subgroup. We denote by $\mathrm{C}_G(H)$ the centralizer, by $\mathrm{N}_G(H)$ the normalizer of H in G, by $\mathrm{Z}(G)$ the center of G. For a prime p, we denote by $\mathrm{O}_p(G)$ the biggest normal p-subgroup of G. Similarly, $\mathrm{O}_{p'}(G)$ is the biggest normal p'-subgroup.

The class function obtained from inducing or restricting another one is shortly denoted by χ_Y, respectively ψ^X, where $Y \leq X$ are finite groups, ψ is a class function of Y and χ one of X. We write $\mathrm{Irr}(G)$ for the set of irreducible (complex) characters of a finite group G, and $\mathrm{Irr}(G \mid \psi)$ for the set of irreducible components of ψ^G whenever $\psi \in \mathrm{Irr}(H)$ and $H \lhd G$. The set of all characters of G is denoted by $\mathrm{Char}(G)$ and $\mathrm{Char}(G \mid \psi)$ is the set of characters whose constituents are $\mathrm{Irr}(G \mid \psi)$. When $\chi \in \mathrm{Char}(G)$, we denote by $\mathrm{Irr}(\chi)$ the set of its irreducible constituents.

1. Clifford theory via projective representations

The aim of Clifford theory is to study how the representation theory of a group is influenced by the one of a normal subgroup and associated quotient groups. More precisely one studies the following situation:

Let $N \lhd G$ be finite groups. From the action of the group G on the normal subgroup N via conjugation, one deduces an action of G on maps on N: for $g \in G$ and $\theta \in \mathrm{Irr}(N)$, one denotes by $\theta^g \in \mathrm{Irr}(N)$ the character defined by

$$\theta^g(n) = \theta(gng^{-1}) \text{ for every } n \in N.$$

If $\theta \in \mathrm{Irr}(N)$, one denotes its stabilizer as $G_\theta := \{g \in G \mid \theta^g = \theta\}$, a subgroup of G also known as inertia group. If a group A acts on the group N we define $\mathrm{Irr}_A(N)$ as the subset of all characters in $\mathrm{Irr}(N)$ that are A-invariant. For any vector space V and any map $\mathcal{D} : N \longrightarrow \mathrm{GL}(V)$, for example given by a linear or projective representation, $\mathcal{D}^g : N \longrightarrow \mathrm{GL}(V)$ is defined by $\mathcal{D}^g(n) = \mathcal{D}(gng^{-1})$ for $n \in N$.

1.A. Clifford theorems and projective representations. Already in the 19th century Frobenius applied the action mentioned above in the study of complex characters. A starting point in this theory is the following fundamental statement.

Theorem 1.1 (Clifford, see [I, 6.2]). *Let $N \lhd G$, $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ a constituent of the restriction χ_N. Then*

$$\chi_N = e \sum_{i=1}^{t} \theta_i$$

for some $e \in \mathbb{N}$ and $\{\theta_1, \dots, \theta_t\}$ the G-orbit containing θ. In particular $t = |G : G_\theta|$.

This motivates the study of the sets $\mathrm{Irr}(G \mid \theta)$ and $\mathrm{Char}(G \mid \theta)$.

Theorem 1.2 (Clifford correspondence, see [I, 6.11]). *Suppose $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ and denote $T := G_\theta$. Then there is a bijection $\mathrm{Irr}(T \mid \theta) \longrightarrow \mathrm{Irr}(G \mid \theta)$ given by induction $\psi \mapsto \psi^G$.*

The above reduces the description of $\mathrm{Irr}(G \mid \theta)$ to the case when $G_\theta = G$.

Theorem 1.3 (Gallagher, see [I, 6.17]). *Suppose $N \lhd G$, $\theta \in \mathrm{Irr}_G(N)$ so that $G_\theta = G$. Assume moreover that θ extends to G, that is $\theta = \chi_N$ for some $\chi \in \mathrm{Irr}(G)$. Then the map $\mathrm{Irr}(G/N) \longrightarrow \mathrm{Irr}(G \mid \theta)$ defined by $\eta \mapsto \chi\eta$ is bijective. (Note that η on the right side is the lift of the character η defined on G/N.) More generally, if $\tau \in \mathrm{Irr}(N)$ with $G = G_\tau$ and $\tau\theta \in \mathrm{Irr}(N)$, then $\eta \mapsto \chi\eta$ defines a bijection*

$$\mathrm{Irr}(G \mid \theta) \longrightarrow \mathrm{Irr}(G \mid \tau\theta).$$

One has to deal however with the general case of an element of $\mathrm{Irr}_G(N)$ that does not extend to G. This is done by means of projective representations.

Definition 1.4. A *(complex) projective representation of G of degree n* is a map $\mathcal{P} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ such that

$$\mathcal{P}(g)\mathcal{P}(h) = \alpha(g,h)\mathcal{P}(gh) \text{ for any } g, h \in G,$$

where $\alpha(g,h) \in \mathbb{C}^\times$. Equivalently $\mathcal{P}$ is required to induce a group morphism $\overline{\mathcal{P}} : G \longrightarrow \mathrm{PGL}_n(\mathbb{C})$. Then $\mathcal{P}$ determines a map $\alpha : G \times G \longrightarrow \mathbb{C}^\times$, called the *factor set of* $\mathcal{P}$. Let $M \in \mathrm{GL}_n(\mathbb{C})$, then $M^{-1}\mathcal{P}M$ defined by $g \mapsto M^{-1}\mathcal{P}(g)M$ is a projective representation with the same factor set as $\mathcal{P}$. Then $\mathcal{P}$ and $M^{-1}\mathcal{P}M$ are called *similar*, written as $\mathcal{P} \sim M^{-1}\mathcal{P}M$.

We say $\mathcal{P}$ is *irreducible* if it is not similar to a projective representation of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

The definition of α implies that the following equality holds

$$\alpha(g_1g_2,g_3)\alpha(g_1,g_2)=\alpha(g_1,g_2g_3)\alpha(g_2,g_3) \text{ for every } g_1,g_2,g_3\in G,$$

see also [I, 11.3]. Maps with this property are called 2-*cocycles*. Together with the naiv multiplication of those maps they form the abelian group $\mathrm{Z}^2(G,\mathbb{C}^\times)$. For every map $\mu: G\to\mathbb{C}^\times$ one can define some $\Delta\mu\in \mathrm{Z}^2(G,\mathbb{C}^\times)$ with $\Delta\mu(g,g')=\mu(g)\mu(g')\mu(gg')^{-1}$. Elements $\alpha\in\mathrm{Z}^2(G,\mathbb{C}^\times)$ that can be written as $\Delta\mu$ are called 2-*coboundaries* and form the group $\mathrm{B}^2(G,\mathbb{C}^\times)$. Relevant for further considerations is the quotient $\mathrm{H}^2(G,\mathbb{C}^\times):=\mathrm{Z}^2(G,\mathbb{C}^\times)/\mathrm{B}^2(G,\mathbb{C}^\times)$, a finite abelian group called the *Schur multiplier* of G.

Remark 1.5 (More on projective representations).

(a) Given a projective representation $\mathcal{P}:G\to\mathrm{GL}_n(\mathbb{C})$ of G and a map $\mu:G\to\mathbb{C}^\times$, one defines $\mu\mathcal{P}$ by $\mu\mathcal{P}(g):=\mu(g)\mathcal{P}(g)$ for every $g\in G$. Note that if $\mathcal{P}$ has factor set α, then $\mu\mathcal{P}$ is a projective representation with factor set $\alpha\,\Delta\mu$.

(b) For a given $\alpha\in\mathrm{Z}^2(G,\mathbb{C}^\times)$, one can form the twisted group algebra $\mathbb{C}^\alpha G$ as the set of linear combinations $\sum_{g\in G}\lambda_g g$ of elements of G with bilinear product $\cdot^\alpha$ satisfying $g\cdot^\alpha g'=\alpha(g,g')gg'$. Then just like linear representations of G are in bijection with representations of the ordinary group algebra, one has a bijection

$$\{\text{projective rep. of } G \text{ with factor set } \alpha\}\overset{1-1}{\longleftrightarrow}\{\text{linear rep. of } \mathbb{C}^\alpha G\}.$$

1.B. Character triples and associated central extensions. The Clifford correspondence introduced above makes it natural to study character triples that are defined in the following way.

Definition 1.6. If $N\lhd G$ and $\theta\in\mathrm{Irr}_G(N)$, then we call (G,N,θ) a *character triple*.

In the study of $\mathrm{Irr}(G\mid\theta)$ projective representations associated to θ are key.

Definition 1.7 (Projective representations associated with (G,N,θ))**.** We say that a projective representation $\mathcal{P}$ of G is *associated with the character triple* (G,N,θ) *or with* θ if the following are both satisfied:

- $\mathcal{P}_N$ is a (linear) representation of N affording θ,
- $\mathcal{P}(ng)=\mathcal{P}(n)\mathcal{P}(g)$ and $\mathcal{P}(gn)=\mathcal{P}(g)\mathcal{P}(n)$ for all $n\in N$, $g\in G$.

According to [I, 11.2] there exists a projective representation to every character triple. Here is a list of useful properties of such projective representations.

Lemma 1.8. *Let $\mathcal{P}$ be a projective representation associated to a character triple (G,N,θ) and let α be the factor set of $\mathcal{P}$. Then*

(a) $1=\alpha(1,1)=\alpha(g,n)=\alpha(n,g)$ *for every $n\in N$ and $g\in G$.*

(b) $\alpha(g_1n_1,g_2n_2)=\alpha(g_1,g_2)$ *for every $n_1,n_2\in N$ and $g_1,g_2\in G$.*

(c) $\mathcal{P}(g)^{-1}\mathcal{P}(n)\mathcal{P}(g)=\mathcal{P}(n^g)$ *for every $n\in N$ and $g\in G$.*

(d) Let $\mathcal{P}':G\longrightarrow\mathrm{GL}_{\theta(1)}(\mathbb{C})$. The following are equivalent:

  (i) $\mathcal{P}'$ is a projective representation associated with (G,N,θ).

  (ii) There exist a matrix $M\in\mathrm{GL}_{\theta(1)}(\mathbb{C})$ and a map $\overline{\mu}:G/N\longrightarrow\mathbb{C}^\times$ such that $\overline{\mu}(1_{G/N})=1$ and $\mathcal{P}'=\mu M\mathcal{P}M^{-1}$, where $\mu:G\longrightarrow\mathbb{C}^\times$ is the lift of $\overline{\mu}$.

Proof. Part (a) follows from the definition of character triples. For the proof of part (b) consider the equation

$$\begin{aligned}\alpha(g_1n_1,g_2n_2)\mathcal{P}(g_1n_1g_2n_2)&=\mathcal{P}(g_1n_1)\mathcal{P}(g_2n_2)=\mathcal{P}(n_1^{g_1^{-1}})\mathcal{P}(g_1)\mathcal{P}(g_2)\mathcal{P}(n_2)=\\&=\alpha(g_1,g_2)\mathcal{P}(n_1^{g_1^{-1}})\mathcal{P}(g_1g_2)\mathcal{P}(n_2)=\\&=\alpha(g_1,g_2)\mathcal{P}(g_1n_1g_2n_2).\end{aligned}$$

The equality in (c) follows from $\mathcal{P}(n)\mathcal{P}(g) = \mathcal{P}(ng) = \mathcal{P}(gg^{-1}ng) = \mathcal{P}(g)\mathcal{P}(n^g)$.

For the proof of part (d), it is clear that whenever $\mathcal{P}'$ is as in (ii), $\mathcal{P}$ is a projective representation associated with (G, N, θ).

To show the converse, let $\mathcal{P}'$ be a projective representation associated with (G, N, θ). Then $\mathcal{P}'_N$ affords θ, and therefore there exists $M \in \mathrm{GL}_{\theta(1)}(\mathbb{C})$ such that $\mathcal{P}'_N = M\mathcal{P}_N M^{-1}$. Without loss of generality we can assume that $\mathcal{P}'_N = \mathcal{P}_N$. By part (c) we have

$$\mathcal{P}(g)\mathcal{P}(n)\mathcal{P}(g)^{-1} = \mathcal{P}'(g)\mathcal{P}(n)\mathcal{P}'(g)^{-1} \text{ for all } n \in N \text{ and } g \in G.$$

Schur's lemma now implies that $\mathcal{P}(g)^{-1}\mathcal{P}'(g)$ is a scalar matrix, so we can write $\mathcal{P}' = \mu\mathcal{P}$ for some unique map $\mu : G \to \mathbb{C}^\times$. This map satisfies $\mu(N) = \{1\}$ since $\mathcal{P}$ and $\mathcal{P}'$ coincide on N. Now the equalities $\mathcal{P}'(gn) = \mathcal{P}'(g)\mathcal{P}'(n)$ and $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ for $g \in G$ and $n \in N$ imply that μ is constant on N-cosets. □

Remark 1.9. In the above situation, let $Z \lhd G$ such that $Z \leq \ker(\theta)$. Then the map $\overline{\mathcal{P}} : G/Z \longrightarrow \mathrm{GL}_{\theta(1)}(\mathbb{C})$ given by $\overline{\mathcal{P}}(gZ) = \mathcal{P}(g)$ (any $g \in G$) is well defined and a projective representation associated with the character triple $(G/Z, N/Z, \overline{\theta})$, where $\overline{\theta}(nZ) = \theta(n)$ for any $n \in N$.

Important in our later consideration is that one can always assume that α has finite order and hence the group $Z_\alpha \leq \mathbb{C}^\times$ generated by the values of α is finite. One can see that as a consequence of the considerations in 1.8(d) together with the isomorphism between $\mathrm{Z}^2(G, \mathbb{C}^\times)$ and $\mathrm{B}^2(G, \mathbb{C}^\times) \oplus \mathrm{H}^2(G, \mathbb{C}^\times)$ from [I, 11.15]. The factor set α of a projective representation $\mathcal{P}$ will allow to relate the general situation for a character triple (G, N, θ) to one where the character extends.

We are here giving an upper bound for the order of α.

Theorem 1.10. *Given a character triple (G, N, θ) and a linear representation $\mathcal{D}$ of N affording θ, there exists a projective representation of G associated with (G, N, θ) such that $\mathcal{P}_N = \mathcal{D}$ and the factor set α of $\mathcal{P}$ satisfies $\alpha(g, h)^{|G|\theta(1)} = 1$ for any $g, h \in G$.*

Proof. For each $\overline{g} = gN$, choose a representation $\mathcal{Q}_{\overline{g}}$ of $\langle g, N\rangle$ extending $\mathcal{D}$. Indeed, since the factor group $\langle g, N\rangle / N$ is cyclic θ extends to $\langle g, N\rangle$ according to [I, 11.22], so that a representation $\mathcal{D}'$ affording that extension has restriction to N of the type $M\mathcal{D}M^{-1}$ for an invertible M. Then one takes $\mathcal{Q}_{\overline{g}} := M^{-1}\mathcal{D}'M$ and defines $\mathcal{P}(g) = \mathcal{Q}_{\overline{g}}(g)$. It is easy to see, using Schur's lemma, that $\mathcal{P}$ is a projective representation.

Since $\mathcal{Q}_{\overline{g}}$ is a linear representation, one has $\det(\mathcal{Q}_{\overline{g}}(g))^{|G|} = 1$. Then the equality defining the factor set α of $\mathcal{P}$ gives $1 = \det(\mathcal{P}(g)\mathcal{P}(h)\mathcal{P}(gh)^{-1})^{|G|} = \alpha(g, h)^{|G|\theta(1)}$. □

The given projective representation $\mathcal{P}$ of G with factor set α allows to construct a central group extension $1 \longrightarrow Z_\alpha \longrightarrow \widehat{G} \twoheadrightarrow G \longrightarrow 1$ and a linear representation $\mathcal{D}$ of $\widehat{G}$ such that $\mathcal{P}$ lifts to a representation $\mathcal{D}$ of $\widehat{G}$. The central extension associated to a factor set on a finite group is obtained by the following construction, see also [Na1, p. 183].

Proposition 1.11. *Let A be an abelian group written multiplicatively and $\alpha : G \times G \longrightarrow A$ a factor set, i.e., one has $\alpha(g_1, g_2)\alpha(g_1g_2, g_3) = \alpha(g_1, g_2g_3)\alpha(g_2, g_3)$ for any $g_1, g_2, g_3 \in G$.*

The set $\Gamma := G \times A$ becomes a group via the multiplication

$$(g, a)(h, b) = (gh, \alpha(g, h)ab) \text{ for all } g, h \in G \text{ and } a, b \in A.$$

Moreover the map $\pi : \Gamma \longrightarrow G$ defined by $(g, a) \mapsto g$ is a central extension of G with kernel $A \cong 1 \times A \leq \mathrm{Z}(\Gamma)$.

This pair (Γ, π) is the *central extension of G given by α* (and A).

Theorem 1.12. *Suppose (G, N, θ) is a character triple and $\mathcal{P}$ an associated projective representation with factor set α. Assume that $Z_\alpha := \langle \alpha(g, h) \mid g, h \in G\rangle \leq \mathbb{C}^\times$ is finite. So that in the central extension $(\widehat{G}, \pi)$ of G given by α the group $\widehat{G}$ is finite.*

a) The groups $N_0 = N\times 1 \cong N$ *and* $Z_0 = 1\times Z_\alpha$ *satisfy* $N_0 \lhd \widehat{G}$, $Z_0 \leq \mathrm{Z}(\widehat{G})$ *and* $\widehat{N} := N_0 \times Z_\alpha \lhd \widehat{G}$

b) The map $\mathcal{D} : \widehat{G} \longrightarrow \mathrm{GL}_{\theta(1)}(\mathbb{C})$ *defined by* $(g,z) \mapsto z\mathcal{P}(g)$ *is a linear representation of* $\widehat{G}$. *Moreover its character* $\tau \in \mathrm{Irr}(\widehat{G})$ *satisfies* $\tau_{N_0} = \theta\circ\pi_{N_0}$ *(, i.e.,* τ *is an extension of* $\theta\circ\pi_{N_0}$*).*

Proof. We observe that $(1,1)$ is the unit element of $\widehat{G}$ since $\alpha(1,1) = \alpha(g,1) = \alpha(1,g) = 1$ for every $g \in G$. Hence $(g,z)^{-1} = (g^{-1}, \alpha(g,g^{-1})^{-1}z^{-1})$. On the other hand $\widehat{N} = N_0 \times Z_0$ since $\alpha(g,n) = \alpha(n,g) = 1$ for all $g \in G$, $n \in N$. This implies the group-theoretic statements.

The map $\mathcal{D}$ is multiplicative since

$$\begin{aligned}\mathcal{D}(g_1,z_1)\mathcal{D}(g_2,z_2) &= z_1z_2\mathcal{P}(g_1)\mathcal{P}(g_2) = z_1z_2\alpha(g_1,g_2)\mathcal{P}(g_1g_2) = \\ &= \mathcal{D}(g_1g_2, \alpha(g_1,g_2)z_1z_2) = \mathcal{D}(g_1g_2,\alpha(g_1,g_2)z_1z_2) \\ &= \mathcal{D}((g_1,z_1)(g_2,z_2)).\end{aligned}$$

Observe that $\mathcal{D}(n,1) = \mathcal{P}(n)$ and hence $\mathcal{P}_{N_0}$ affords $\theta\circ\pi_{N_0}$ as character. □

This construction now allows to relate the characters of G above θ with those of a quotient of $\widehat{G}$.

Corollary 1.13. *Assume the situation of Theorem 1.12. Let* $G^* = \widehat{G}/N_0$, λ *the irreducible constituent of* τ_{Z_0} *and* $\widehat{\lambda}$ *the character of* $(N_0\times Z_0)/N_0$ *induced by* λ. *Then there exists a bijection*

$$\begin{aligned}\mathrm{Irr}(G\mid\theta) &\longrightarrow \mathrm{Irr}(G^*\mid \overline{\lambda}^{-1}) \\ (\tau\eta)\circ\pi &\longmapsto \overline{\eta}\end{aligned}$$

where $\overline{\eta}$ *lifts to* η.

Proof. Note that $\tau_{\widehat{N}} = \theta_0\times\lambda$ where $\theta_0 = \theta\circ\pi_{N_0}$ and $\tau_{\widehat{N}}(1_{N_0}\times\lambda^{-1}) = \theta_0\times 1_{Z_0} \in \mathrm{Irr}(\widehat{N})$. Hence by Theorem 1.3, we get a bijection $\mathrm{Irr}(\widehat{G}\mid\theta_0\times 1_{Z_0}) \to \mathrm{Irr}(\widehat{G}\mid 1_{N_0}\times\lambda^{-1})$.

Using the statement of Gallagher one gets the following maps between the various character sets:

$$\begin{tikzcd}
\mathrm{Irr}(G\mid\theta) \arrow[r, leftrightarrow, "\text{lift}"] & \mathrm{Irr}(\widehat{G}\mid\theta_0\times 1_{Z_0}) \arrow[r, "1-1"] & \mathrm{Irr}(\widehat{G}\mid 1_{N_0}\times\lambda^{-1}) \arrow[r] & \mathrm{Irr}(G^*\mid\overline{\lambda}^{-1}) \\
(\tau\eta)\circ\pi^{-1} \arrow[r, mapsto] & \tau\eta \arrow[r, mapsto] & \eta \arrow[r, mapsto] & \overline{\eta}.
\end{tikzcd}$$

□

Notation 1.14. For a 2-cocycle $\alpha : G\times G \longrightarrow \mathbb{C}^\times$ on G, one defines $\mathrm{Proj}(G\mid\alpha)$ as the set of projective representations of G with factor set α. For two 2-cocycles $\alpha,\beta : G\times G\longrightarrow\mathbb{C}^\times$ the tensor product of matrices gives a natural map

$$\begin{aligned}\mathrm{Proj}(G\mid\alpha)\times\mathrm{Proj}(G\mid\beta) &\longrightarrow \mathrm{Proj}(G\mid\alpha\beta) \\ (\mathcal{P},\mathcal{Q}) &\longmapsto \mathcal{P}\otimes\mathcal{Q},\end{aligned}$$

where $\mathcal{P}\otimes\mathcal{Q}$ is given by $(\mathcal{P}\otimes\mathcal{Q})(g) = \mathcal{P}(g)\otimes\mathcal{Q}(g)$ for $g\in G$.

Recall from Definition 1.4 two projective representations $\mathcal{P}_1$, $\mathcal{P}_2 \in \mathrm{Proj}(G\mid\alpha)$ are said to be similar if for some $M\in\mathrm{GL}_n(\mathbb{C})$ one has $\mathcal{P}_1(g) = M\mathcal{P}_2(g)M^{-1}$ $(g\in G)$. Accordingly similar projective representations have the same dimension n.

Theorem 1.15 (Clifford). *Let* (G,N,θ) *be a character triple and* $\mathcal{P}$ *a projective representation of* G *associated to it and with factor set* α. *Let* $\mathrm{Rep}(G\mid\theta)$ *be the set of representations of* G *affording a character in* $\mathrm{Char}(G\mid\theta)$. *Then the map*

$$\begin{aligned}\mathrm{Proj}(G/N\mid\alpha^{-1}) &\longrightarrow \mathrm{Rep}(G\mid\theta) \\ \mathcal{Q} &\mapsto \mathcal{Q}\otimes\mathcal{P}\end{aligned}$$

is injective and:

(a) For every $\chi \in \operatorname{Irr}(G \mid \theta)$ there exists $\mathcal{Q} \in \operatorname{Proj}(G/N \mid \alpha^{-1})$ such that $\mathcal{Q} \otimes \mathcal{P}$ is a linear representation affording χ.

(b) In the above situation $\mathcal{Q} \otimes \mathcal{P}$ is irreducible if and only if $\mathcal{Q}$ is irreducible.

(c) $\mathcal{Q} \otimes \mathcal{P}$ and $\mathcal{Q}' \otimes \mathcal{P}$ are similar if and only if $\mathcal{Q}$ and $\mathcal{Q}'$ are similar.

Proof. This follows from [Na1, 8.16, 8.18] and [NT, 3.5.7]. □

2. Centrally ordered character triples

In this section we discuss in detail how the Clifford theory of two characters can be compared when the relevant factor groups are naturally isomorphic. Namely, we have character triples (G, N, θ) and (H, M, φ) such that $H \leq G$ where $G = HN$ and $M = H \cap N$.

We introduce the order relations $\geq$ (Definition 2.1 below) and $\geq_c$ (Definition 2.7) which in this context of group theoretical dominance expresses a rather tight equivalence of the Clifford theories on both sides. The Butterfly Theorem 2.16 shows that $\geq_c$ allows a certain freedom in the choice of the overgroups. Indeed more than the factor group G/N it is the induced subgroup of $\operatorname{Aut}(N)$ that actually matters.

2.A. Strong isomorphisms of character triples, central quotients.

Definition 2.1 (A first order relation)**.** Let (G, N, θ) and (H, M, φ) be two character triples. We write

$$(G, N, \theta) \geq (H, M, \varphi)$$

if

(1) $G = NH$ and $M = N \cap H$.

(2) There exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ associated with (G, N, θ) and (H, M, φ) such that their factor sets α and α', respectively satisfy $\alpha' = \alpha_{H \times H}$.

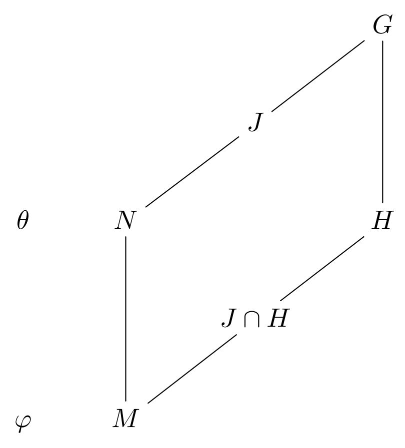
Then we say that $(G, N, \theta) \geq (H, M, \varphi)$ is *given by* $(\mathcal{P}, \mathcal{P}')$.

Note that in the situation above the pair $(\mathcal{P}, \mathcal{P}')$ is not unique. The relevance of the relation defined above comes essentially from the following.

Theorem 2.2. *Let $(G, N, \theta) \geq (H, M, \varphi)$ be given by $(\mathcal{P}, \mathcal{P}')$. In this case, for every subgroup J with $N \leq J \leq G$, there is a well-defined bijection*

$$\begin{aligned} \sigma_J : \operatorname{Char}(J \mid \theta) &\longrightarrow \operatorname{Char}(J \cap H \mid \varphi) \\ \operatorname{Tr}(\mathcal{Q} \otimes \mathcal{P}_J) &\mapsto \operatorname{Tr}(\mathcal{Q}_{J \cap H} \otimes \mathcal{P}_{J \cap H}) \end{aligned}$$

with $\sigma_J(\operatorname{Irr}(J \mid \theta)) = \operatorname{Irr}(J \cap H \mid \varphi)$.



Proof. Let α be the factor set of $\mathcal{P}$. By Theorem 1.15 there exists, for every $\chi \in \mathrm{Char}(G \mid \theta)$, some $\mathcal{Q} \in \mathrm{Proj}(G \mid \alpha^{-1})$ such that $\mathcal{Q} \otimes \mathcal{P}_J$ affords χ. Here $\mathcal{Q}$ is unique up to similarity so $\mathrm{Tr}(\mathcal{Q}_{J\cap H} \otimes \mathcal{P}_{J\cap H})$ is well-defined (using that the two terms in the tensor product have inverse factor sets). This implies that σ_J is well-defined. If moreover χ is irreducible, then $\mathcal{Q}$ is irreducible and $\sigma_J(\chi)$ is irreducible. □

Naturally this comes up in the following situation.

Proposition 2.3. *Assume (G, N, θ) and (H, M, φ) are two character triples with $H \leq G = NH$, $M = N \cap H$ and $\varphi = \theta_M$. Then $(G, N, \theta) \geq (H, M, \varphi)$.*

Proof. Let $\mathcal{P}$ be a projective representation of G associated with the character triple (G, N, θ). Then it is clear from Definition 1.7 and our hypotheses that $\mathcal{P}' := \mathcal{P}_H$ is a projective representation associated with (H, M, φ). Let α, α' be the factor sets of $\mathcal{P}$, $\mathcal{P}'$ respectively. Then it is clear from the definition of $\mathcal{P}'$ that $\alpha' = \alpha_{H\times H}$. □

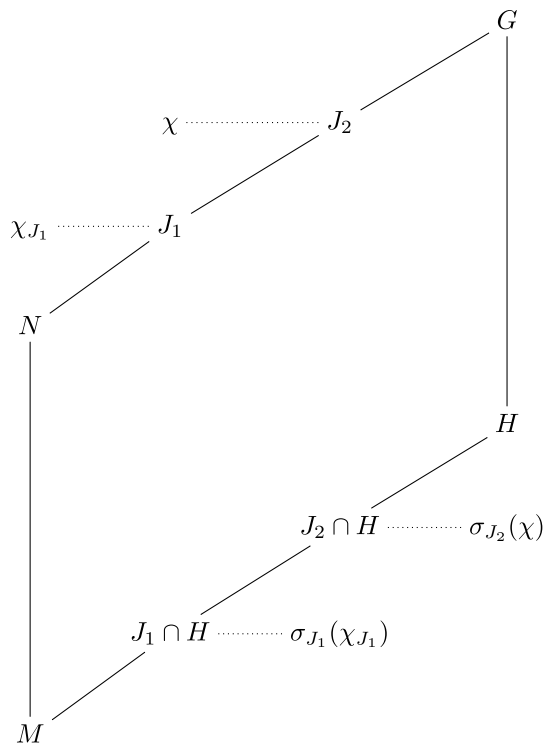
Although the following result is not explicitly used in the sequel it gives additional information on the nature of the map σ_J.

Corollary 2.4 (More properties of the map σ_J)**.** *The map σ_J from Theorem 2.2 gives a strong isomorphism of character triples in the sense of [I, Problem 11.13]. Namely, for any $J_1 \leq J_2 \leq G$:*

(a) $\sigma_{J_2}(\chi)_{J_1\cap H} = \sigma_{J_1}(\chi_{J_1})$ for every $\chi \in \mathrm{Char}(J_2 \mid \theta)$,

(b) $\sigma_{J_2}(\chi\beta) = \sigma_{J_2}(\chi)\beta_{J_2\cap H}$ for every $\beta \in \mathrm{Char}(J_2/N)$,

(c) $\sigma_{J_2}(\chi)^h = \sigma_{J_2}(\chi^h)$ for every $h \in H$.



Proof. For the proof of the first part let $\mathcal{Q} \in \mathrm{Proj}(J_2/N \mid \alpha^{-1}_{J_2/N\times J_2/N})$ such that $\mathcal{Q} \otimes \mathcal{P}_{J_2}$ affords χ. Then $(\sigma_{J_2}(\chi))_{J_1\cap H}$ is the character of $(\mathcal{Q}_{J_2\cap H} \otimes \mathcal{P}'_{J_2\cap H})_{J_1\cap H} = \mathcal{Q}_{J_1\cap H} \otimes \mathcal{P}'_{J_1\cap H}$. Hence $\sigma_{J_1}(\chi_{J_1}) = (\sigma_{J_2}(\chi))_{J_1\cap H}$.

For the proof of the equation in (b) let $\mathcal{Q}'$ be a linear representation affording β. Then $(\mathcal{Q}' \otimes \mathcal{Q}) \otimes \mathcal{P}_{J_2}$ affords $\chi\beta$. On the other hand $(\mathcal{Q}' \otimes \mathcal{Q}) \otimes \mathcal{P}'_{J_2 \cap H}$ affords $\sigma_{J_2}(\chi\beta)$. Hence $\mathcal{Q}'_{J_2\cap H} \otimes (\mathcal{Q}_{J_2\cap H} \otimes \mathcal{P}'_{J_2\cap H})$ is a representation with character $\beta_{J_2\cap H}\sigma_{J_2}(\chi)$.

Note that $\mathcal{P}^h$ and $\mathcal{P}$ are projective representations associated with θ. According to Lemma 1.8(d) the projective representations have to satisfy $\mathcal{P}^h \sim \mu\mathcal{P}$ for some map $\mu : J_2/N \to \mathbb{C}^\times$. Analogously $(\mathcal{P}')^h \sim \mu'\mathcal{P}'$ for some map $\mu' : (J_2 \cap H)/M \to \mathbb{C}^\times$. For every $g \in G$ the definition of $\mathcal{P}^h$ implies

$$\mathcal{P}^h(g) = \mathcal{P}(g^{h^{-1}}) = \alpha(h, g)\alpha(hg, h^{-1})\mathcal{P}(h)\mathcal{P}(g)\mathcal{P}(h^{-1}).$$

By standard computation one obtains $\mathcal{P}(h^{-1}) = \alpha(h^{-1}, h)\mathcal{P}(h)^{-1}$. Hence

$$\mu(g) = \alpha(h, g)\alpha(hg, h^{-1})\alpha(h^{-1}, h).$$

The analogous computation for $(\mathcal{P}')^h$ shows that

$$\mu'(g) = \alpha'(h, g)\alpha'(hg, h^{-1})\alpha'(h^{-1}, h),$$

where $g \in J_2$ and α' is the factor set of $\mathcal{P}'$. Since $\alpha_{H\times H} = \alpha'$ we see that $\mu_{J_2\cap H} = \mu'$.

Let $\mathcal{Q} \in \mathrm{Proj}(J_2/N \mid \alpha_{J_2}^{-1})$ such that $\mathcal{Q}\otimes\mathcal{P}_{J_2}$ affords χ. Hence χ^h is the character of $\mathcal{Q}^h\otimes\mathcal{P}_{J_2}^h \sim \mathcal{Q}^h \otimes \mu\mathcal{P}_{J_2} = (\mu\mathcal{Q}^h) \otimes \mathcal{P}_{J_2}$. By definition $\sigma_{J_2}(\chi)$ is the character of $\mathcal{Q}_{J_2\cap H} \otimes \mathcal{P}'_{J_2}$ and $\sigma_{J_2}(\chi^h)$ is the character of $(\mu_{J_2\cap H}\mathcal{Q}^h_{J_2\cap H}) \otimes \mathcal{P}'_{J_2}$. On the other hand

$$(\mathcal{Q}_{J_2\cap H})^h \otimes {\mathcal{P}'_{J_2}}^h \sim (\mathcal{Q}_{J_2\cap H})^h \otimes \mu'\mathcal{P}'_{J_2} = (\mu'\mathcal{Q}_{J_2\cap H})^h \otimes \mathcal{P}'_{J_2}$$

affords $\sigma_{J_2}(\chi)^h$. Since μ and μ' coincide on $J_2 \cap H$ this proves the claimed equality. □

Whenever two character triples satisfy $(G, N, \theta) \geq (H, M, \varphi)$ the pair $(\mathcal{P}, \mathcal{P}')$ and the bijections σ_J defined by $(\mathcal{P}, \mathcal{P}')$ are not uniquely determined. The choice of the pair is limited in the following way.

Proposition 2.5. *Let $(G, N, \theta) \geq (H, M, \varphi)$ be given by $(\mathcal{P}, \mathcal{P}')$. Then the following are equivalent:*

- *$(\mathcal{Q}, \mathcal{Q}')$ is another pair giving $(G, N, \theta) \geq (H, M, \varphi)$,*
- *there exists maps $\mu : G/N \to \mathbb{C}^\times$ and $\mu' : H/M \to \mathbb{C}^\times$ such that $\mathcal{Q} \sim \mu\mathcal{P}$, $\mathcal{Q}' \sim \mu'\mathcal{P}'$ and $\Delta\mu = \Delta\mu'$.*

Then, for a subgroup J with $N \leq J \leq G$ the bijections σ_J and σ'_J defined using $(\mathcal{P}, \mathcal{P}')$ and $(\mathcal{Q}, \mathcal{Q}')$, respectively satisfy

$$\sigma_J(\chi) = \sigma'_J(\chi)\lambda_{J\cap H} \text{ for every } \chi \in \mathrm{Char}(J \mid \theta),$$

where the map $\lambda := (\mu_{H/M})^{-1}\mu'$ is a linear character of H/M.

Proof. We make use of Lemma 1.8 and combine it with Definition 2.1. As projective representations associated with θ and φ the projective representations $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy $\mathcal{Q} \sim \mu\mathcal{P}$ and $\mathcal{Q}' \sim \mu'\mathcal{P}'$ for some maps $\mu : G/N \longrightarrow \mathbb{C}^\times$ and $\mu' : H/M \longrightarrow \mathbb{C}^\times$. Their factor sets are then $\alpha\,\Delta\mu$ and $\alpha'\,\Delta\mu'$, where α and α' denote the factor sets of $\mathcal{P}$ and $\mathcal{P}'$, respectively. These factor sets satisfy the required property $(\alpha\,\Delta\mu)_{H\times H} = (\alpha'\,\Delta\mu')_{H\times H}$ if and only if $\Delta\mu = \Delta\mu'$. Accordingly the map λ is a linear character of H/M. The rest of the statement follows from the construction of σ_J. □

Remark 2.6. (a) If $(G, N, \theta) \geq (H, M, \varphi)$, the characters above θ and φ can be compared via the bijection from Theorem 2.2. In this sense one controls the Clifford theory of θ and φ.

(b) On the other given $(G,N,\theta) \geq (H,M,\varphi)$ and a group $Z \lhd G$ with $Z \cap N = \{1\}$ it is not possibly to compare the Clifford theory of the characters $\overline{\theta}$ and $\overline{\varphi}$ of NZ/Z and MZ/Z given by θ and φ. The sets $\mathrm{Irr}(G/Z \mid \overline{\theta})$ and $\mathrm{Irr}(HZ/Z \mid \overline{\theta})$ might have different cardinalities.

Let $\mathcal{P}$ be associated to the character triple (G,N,θ) and let $c \in \mathrm{C}_G(N)$. Then $\mathcal{P}(c)$ is a scalar matrix by Schur's lemma. Therefore $\mathcal{P}$ defines a map $\zeta : \mathrm{C}_G(N) \longrightarrow \mathbb{C}^\times$ by $\mathcal{P}(c) = \zeta(c)\,\mathrm{id}_{\theta(1)}$. Then $\zeta(cz) = \zeta(c)\lambda(z)$ for every $z \in \mathrm{Z}(N)$ where λ is a constituent of $\theta_{\mathrm{Z}(N)}$. So ζ is a projective representation of $\mathrm{C}_G(N)$ associated with the character triple $(\mathrm{C}_G(N), \mathrm{Z}(N), \lambda)$.

Definition 2.7. We write
$$(G,N,\theta) \geq_{\mathrm{c}} (H,M,\varphi)$$
if

(a') $(G,N,\theta) \geq (H,M,\varphi)$

(b') $\mathrm{C}_G(N) \leq H$

(c') there exist $(\mathcal{P},\mathcal{P}')$ giving $(G,N,\theta) \geq (H,M,\varphi)$ (see Definition 2.1) such that for any $c \in \mathrm{C}_G(N)$, there exists $\zeta \in \mathbb{C}^\times$ with $\mathcal{P}(c) = \zeta\,\mathrm{id}_{\theta(1)}$ and $\mathcal{P}'(c) = \zeta\,\mathrm{id}_{\varphi(1)}$.

Then we call $(\mathcal{P},\mathcal{P}')$ *associated with* $(G,N,\theta) \geq_{\mathrm{c}} (H,M,\varphi)$ in the above situation, or that $(G,N,\theta) \geq_{\mathrm{c}} (H,M,\varphi)$ is given by $(\mathcal{P},\mathcal{P}')$.

For the later use we summarize the conditions that have to be satisfied for two character triples to satisfy the above order relation.

Remark 2.8. Let (G,N,θ) and (H,M,φ) be two character triples. Then
$$(G,N,\theta) \geq_{\mathrm{c}} (H,M,\varphi)$$
if

(a) $H \leq G$, $G = NH$ and $M = N \cap H$,

(b) there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ respectively associated with $(G.N,\theta)$ and (H,M,φ) such that their factor sets α and α' satisfy $\alpha' = \alpha_{H\times H}$,

(c) $\mathrm{C}_G(N) \leq H$,

(d) the projective representations $\mathcal{P}$ and $\mathcal{P}'$ can be chosen such that for every $c \in \mathrm{C}_G(N)$, there exists $\zeta \in \mathbb{C}^\times$ with $\mathcal{P}(c) = \zeta\,\mathrm{id}_{\theta(1)}$ and $\mathcal{P}'(c) = \zeta\,\mathrm{id}_{\varphi(1)}$.

A first example is as follows.

Lemma 2.9. *Suppose $N \lhd G$ with $N\mathrm{C}_G(N) = G$. For $Z = \mathrm{Z}(N)$, $\lambda \in \mathrm{Irr}(Z)$ and $\theta \in \mathrm{Irr}(N \mid \lambda)$ we have $(G,N,\theta) \geq_{\mathrm{c}} (\mathrm{C}_G(N), Z, \lambda)$.*

Proof. Let $\mathcal{P}$ be a projective representation of G associated with (G,N,θ). Take $\zeta : \mathrm{C}_G(N) \longrightarrow \mathbb{C}^\times$ with $\mathcal{P}(c) = \zeta(c)\,\mathrm{id}_{\theta(1)}$ for all $c \in \mathrm{C}_G(N)$. Then ζ is a projective representation associated with λ. Therefore $(\mathcal{P},\zeta)$ gives $(G,N,\theta) \geq_{\mathrm{c}} (\mathrm{C}_G(N), Z, \lambda)$. □

Key to later applications is the following character correspondence.

Lemma 2.10. *If $(G,N,\theta) \geq_{\mathrm{c}} (H,M,\varphi)$, then the associated maps σ_J from Theorem 2.2 satisfy*
$$\sigma_J(\mathrm{Irr}(J \mid \theta) \cap \mathrm{Irr}(J \mid \kappa)) \subseteq \mathrm{Irr}(J \cap H \mid \kappa)$$
for every $\kappa \in \mathrm{Irr}(\mathrm{C}_G(N) \cap J)$ and every J with $N \leq J \leq G$.

Proof. This follows from Definition 2.7. □

Proposition 2.11. *Let $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$ be given by $(\mathcal{P}, \mathcal{P}')$. Then the following are equivalent:*

- *$(\mathcal{Q}, \mathcal{Q}')$ is another pair giving $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$,*
- *then there exist maps $\mu : G/N \longrightarrow \mathbb{C}^\times$ and $\mu' : H/M \longrightarrow \mathbb{C}^\times$ such that $\mathcal{Q} \sim \mu\mathcal{P}$, $\mathcal{Q}' \sim \mu'\mathcal{P}'$, $\Delta\mu = \Delta\mu'$ and $\mu_{\mathrm{C}_G(N)/\mathrm{Z}(N)} = \mu'_{\mathrm{C}_G(N)/\mathrm{Z}(N)}$.*

Proof. The proof follows the arguments of Proposition 2.5. Considering the values of the projective representations on $\mathrm{C}_G(N)$ implies that $\mu_{\mathrm{C}_G(N)/\mathrm{Z}(N)} = \mu'_{\mathrm{C}_G(N)/\mathrm{Z}(N)}$ has to be satisfied. □

Often projective representations associated with a character or a character triple are studied via their factors set or an element in $\mathrm{H}^2(G/N, \mathbb{C}^\times)$. The considerations here take more invariants into account.

Remark 2.12. The projective representations $\mathcal{P}$ and $\mathcal{P}'$ associated with the characters have to coincide in two invariants associated with them: the factor set $\alpha \in \mathrm{Z}^2(G/N, \mathbb{C}^\times)$ and a map $\zeta \in \mathrm{Map}(\mathrm{C}_G(N), \mathbb{C}^\times)$, where $\mathrm{Map}(\mathrm{C}_G(N), \mathbb{C}^\times)$ is the set of all maps from $\mathrm{C}_G(N)$ to $\mathbb{C}^\times$.

The projective representation $\mathcal{P}$ associated with a character triple (G, N, θ) is not uniquely determined by that. By Lemma 1.8(d) $\mathcal{P}$ can be replaced by a projective representation similar to $\mu\mathcal{P}$, where $\mu : G/N \to \mathbb{C}^\times$. Let us denote by α the factor set of $\mathcal{P}$ and by $\zeta \in \mathrm{Map}(\mathrm{C}_G(N), \mathbb{C}^\times)$ the map given by $\mathcal{P}$. Now the factor set of $\mu\mathcal{P}$ is $\alpha\,\Delta\mu$ and restricting to $\mathrm{C}_G(N)$ one obtains a map $\zeta(\mu_{\mathrm{C}_G(N)/\mathrm{Z}(N)})$. We define on $\mathrm{Z}^2(G, \mathbb{C}^\times) \times \mathrm{Map}(\mathrm{C}_G(N)/\mathrm{Z}(N), \mathbb{C}^\times)$ an equivalence relation by identifying (α, ζ) with $(\alpha\Delta\mu, \zeta\mu_{\mathrm{C}_G(N)})$ for any $\mu \in \mathrm{Map}(G/N, \mathbb{C}^\times)$. We denote the set of equivalence classes by $\widehat{\mathrm{H}}^2(G/N, \mathrm{C}_G(N), \mathbb{C}^\times)$.

By this definition θ determines a unique element $[\theta]_c$ of $\mathrm{Z}^2(G, \mathbb{C}^\times) \times \mathrm{Map}(\mathrm{C}_G(N)/\mathrm{Z}(N), \mathbb{C}^\times)$ up to equivalence. If (G, N, θ) and (H, M, φ) satisfy the conditions (a) and (c) from Remark 2.8 we can compare the elements $[\theta]_{\mathrm{c}}$ and $[\varphi]_{\mathrm{c}}$, since the sets $\widehat{\mathrm{H}}^2(G/N, \mathrm{C}_G(N), \mathbb{C}^\times)$ and $\widehat{\mathrm{H}}^2(H/M, \mathrm{C}_G(N), \mathbb{C}^\times)$ can be identified.

Proposition 2.13. *We have $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$ if and only if the group theoretic conditions 2.8(a) and (c) hold and $[\theta]_{\mathrm{c}} = [\varphi]_{\mathrm{c}}$ in $\widehat{\mathrm{H}}^2(H/M, \mathrm{C}_G(N), \mathbb{C}^\times)$.*

Proposition 2.14 (Going to quotients, I)**.** *Let us assume $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$. Let $Z \lhd G$ with $Z \cap N = \{1\}$ (therefore $Z \leq \mathrm{C}_G(N) \leq H$). Then*

$$(G/Z, NZ/Z, \overline{\theta}) \geq_{\mathrm{c}} (H/Z, MZ/Z, \overline{\varphi}).$$

where $\overline{\theta}$, $\overline{\varphi}$ are the characters of NZ/Z, MZ/Z that are induced by θ and φ via the canonical isomorphism between the groups.

Proof. Let $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$ be given by the projective representations $(\mathcal{P}, \mathcal{P}')$. Any projective representation $\overline{\mathcal{P}}_1$ of G/Z associated with $\overline{\theta}$ lifts to a projective representation $\mathcal{P}_1$ associated with θ which is constant on Z-cosets and $\mathcal{P}_{1N} = \mathcal{P}$. Let $\mu : G \longrightarrow \mathbb{C}^\times$ such that $\mu\mathcal{P} = \mathcal{P}_1$. Then $(\mathcal{P}_1, \mu_H\mathcal{P}'_1)$ satisfies all conditions from Definition 2.7 and $(G/Z, NZ/Z, \overline{\theta}) \geq_{\mathrm{c}} (H/Z, MZ/Z, \overline{\varphi})$. □

Note that the above actually needs the relation $\geq_{\mathrm{c}}$ and would not hold with the weaker relation $\geq$ of Definition 2.1.

Lemma 2.15. *Let (G, N, θ) and (H, M, φ) be two character triples with $G = NH$, $M = H \cap N$ and $\mathrm{C}_G(N) \leq H$. Assume that θ extends to some character $\widetilde{\theta}$ of G. Then the following are equivalent:*

(i) $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$

(ii) there exists some extension $\widetilde{\varphi}$ of φ to H such that $\mathrm{Irr}(\widetilde{\varphi}_{\mathrm{C}_G(N)}) = \mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(N)})$.

Proof. Whenever $(G, N, \theta) \geq_c (H, M, \varphi)$ holds the induced bijection $\sigma_G : \mathrm{Char}(G \mid \theta) \to \mathrm{Char}(H \mid \varphi)$ maps $\widetilde{\theta}$ to an extension $\widetilde{\varphi}$ of φ. According to Lemma 2.10 this character satisfies $\mathrm{Irr}(\widetilde{\varphi}_{\mathrm{C}_G(N)}) = \mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(N)})$.

Conversely, let $\mathcal{P}$ be a representation with character $\widetilde{\theta}$ and $\mathcal{P}'$ a representation with character $\widetilde{\varphi}$. Note that here $\mathcal{P}$ and $\mathcal{P}'$ are chosen to be (true) representations, not only projective representations. Then the pair $(\mathcal{P}, \mathcal{P}')$ has the required properties. Hence $(G, N, \theta) \geq_c (H, M, \varphi)$. □

2.B. Building ordered character triples. The nature of the order relation $\geq_c$ is revealed by the following remarkable statement. The main idea is that $(G, N, \theta) \geq_c (H, M, \varphi)$ depends on the characters θ and φ together with the automorphisms of G induced on N.

This statement and its importance crystallized later in the context of Dade's conjecture [S17] but related considerations were already present in [IMN07].

Theorem 2.16 (Butterfly theorem)**.** *Suppose $(G_1, N, \theta) \geq_c (H_1, M, \varphi)$ and $N \lhd G_2$. Assume now that via the canonical morphism $\epsilon_i : G_i \longrightarrow \mathrm{Aut}(N)$ $(i = 1, 2)$ we have $\epsilon_1(G_1) = \epsilon_2(G_2)$. Let $H_2 = \epsilon_2^{-1}\epsilon_1(H_1)$. Then*

$$(G_2, N, \theta) \geq_c (H_2, M, \varphi).$$

The group theoretical picture is as follows

$$\begin{tikzcd}[arrows=dash]
G_2 \arrow[dr] \arrow[dd] & & G_1 \arrow[dl] \arrow[dd] \\
& N\ \theta \arrow[dd] & \\
\epsilon_2^{-1}\epsilon_1(H_1) = H_2 \arrow[dr] & & H_1 \arrow[dl] \\
& M\ \varphi &
\end{tikzcd}$$

Proof. We successively verify the conditions listed in Remark 2.8. By the assumption $\mathrm{C}_{G_2}(N) = \ker \epsilon_2 \subseteq \epsilon_2^{-1}\epsilon_1 H_1 = H_2$ we have condition (c) of Remark 2.8.

About $G_2 = NH_2$. Let $x_2 \in G_2$. Then there exists $x_1 \in G_1$ such that $\epsilon_2(x_2) = \epsilon_1(x_1)$. Because of $G_1 = NH_1$, there exists $n_1 \in N$ and $h_1 \in H_1$ such that $x_1 = n_1 h_1$. The element $n_1^{-1}x_2 \in G_2$ is well-defined and $\epsilon_2(n_1^{-1}x_2) = \epsilon_2(n_1)^{-1}\epsilon_1(x_1) = \epsilon_1(h_1)$ so that $n_1^{-1}x_2 \in \epsilon_2^{-1}\epsilon_1(H) = H_2$. Hence condition 2.8(a).

About $M = N \cap H_2$. We have $N \cap H_2 = N \cap \epsilon_2^{-1}(\epsilon_1(H_1)) \supseteq M$ since $\epsilon_1(H_1) \leq \epsilon_1(M)$.

On the other hand $N \cap H_2 \leq M$ since for every $n \in N \cap H_2$ we have $\epsilon_2(n) = \epsilon_1(h)$ for some $h \in H$. Therefore $nh^{-1} \in \mathrm{C}_{G_1}(N) \leq H_1$ and $n \in N \cap H_1 = M$. So we have conditions 2.8(a) and 2.8(c).

Observe $(G_2)_\theta = G_2$ and $(H_2)_\varphi = H_2$.

Since $(\mathcal{P}_1, \mathcal{P}_1')$ gives $(G_1, N, \theta) \geq_c (H_1, M, \varphi)$ we have

(a) the corresponding factor sets α_1 and α_1' satisfy $\alpha_{1 H_1 \times H_1} = \alpha_1'$,

(b) there is $\zeta : \mathrm{C}_G(N) \longrightarrow \mathbb{C}^\times$ such that $\mathcal{P}_1(c) = \zeta(c)\,\mathrm{id}_{\theta(1)}$ and $\mathcal{P}_1'(c) = \zeta(c)\,\mathrm{id}_{\varphi(1)}$ for any $c \in \mathrm{C}_G(N)$.

Let $1 \in \mathcal{T}_1 \subseteq H_1$ be a complete set of the $M\mathrm{C}_{G_1}(N)$-cosets in H_1. Then, remembering that $G_1 = NH_1$, $\mathcal{T}_1$ represents all $N\mathrm{C}_{G_1}(N)$-cosets in G_1. Choose now for every $t \in \mathcal{T}_1$ an element $\widehat{t} \in H_2$ with $\epsilon_1(t) = \epsilon_2(\widehat{t})$ (remember that $\widehat{1} = 1$). Those elements form the set $\mathcal{T}_2 \subseteq H_2$. We thus get a representative system of all $M\mathrm{C}_{G_2}(N)$ cosets in H_2.

Let ν an irreducible constituent of Let $\widehat{\mu} : \mathrm{C}_{G_2}(N) \longrightarrow \mathbb{C}^\times$ be a map such that

$$(*) \qquad \widehat{\mu}(cz) = \widehat{\mu}(c)\nu(z)$$

for all $c \in \mathrm{C}_{G_2}(N)$ and $z \in \mathrm{Z}(N)$. Define

$$\mathcal{P}_2 : G \longrightarrow \mathrm{GL}_{\theta(1)}(\mathbb{C}) \text{ and } \mathcal{P}_2' : H_2 \longrightarrow \mathrm{GL}_{\varphi(1)}(\mathbb{C})$$

by $\mathcal{P}_2(\widehat{t}nc) = \mathcal{P}_1(t)\mathcal{P}_1(n)\widehat{\mu}(c)$ for $\widehat{t} \in \mathcal{T}_2$, $n \in N$, and $c \in \mathrm{C}_{G_2}(N)$, and by $\mathcal{P}_2'(\widehat{t}mc) = \mathcal{P}_1'(t)\mathcal{P}_1'(m)\widehat{\mu}(c)$ for $\widehat{t} \in \mathcal{T}_2$, $m \in M$, and $c \in \mathrm{C}_{G_2}(N)$.

In order to prove that $(\mathcal{P}_2, \mathcal{P}_2')$ gives $(G_2, N, \theta) \geq_c (H_2, M, \varphi)$ we verify in the remainder of the proof the following: points

(i) this is well-defined

(ii) $\mathcal{P}_2$ and $\mathcal{P}_2'$ are projective representations associated to θ and φ

(iii) the factor sets of $\mathcal{P}_2$ and $\mathcal{P}_2'$ satisfy condition 2.8(b) and

(iv) condition 2.8(d) on $(\mathcal{P}_2)_{\mathrm{C}_{G_2}(N)}$ and $(\mathcal{P}_2')_{\mathrm{C}_{G_2}(N)}$ holds.

Note that (iv) is clear from the construction.

About (i). By the choice of $\mathcal{T}_2$, every element $g \in G_2$ can be written as $g = \widehat{t}nc$ with $\widehat{t} \in \mathcal{T}_2$, $n \in N$, and $c \in \mathrm{C}_{G_2}(N)$. Here $\widehat{t} \in \mathcal{T}_2$ is unique while n and c are unique only up to simultaneous multiplication with inverse elements of $\mathrm{Z}(N)$. Then by definition of $\widehat{\mu}$ and the condition (*), this implies that $\mathcal{P}_2$ and $\mathcal{P}_2'$ are well-defined.

About (ii). $(\mathcal{P}_2)_N = \mathcal{P}_1$ and $(\mathcal{P}_2')_M = \mathcal{P}_1'$ are linear representations affording θ and φ. For every $g \in G_2$ and $n' \in N$ then $\mathcal{P}_2(gn') = \mathcal{P}_2(g)\mathcal{P}_2(n')$ by definition. Writing $g = \widehat{t}nc$ we also have

$$\begin{aligned} \mathcal{P}_2(n'g) = \mathcal{P}_2(\widehat{t}(n')^{\widehat{t}}nc) &= \mathcal{P}_1(t)\mathcal{P}_1((n')^t n)\widehat{\mu}(c) = \mathcal{P}_1(t(n')^t n)\widehat{\mu}(c) \\ &= \mathcal{P}_1(n'tn)\widehat{\mu}(c) = \mathcal{P}_1(n')\mathcal{P}_1(tn) = \mathcal{P}_1(n')\mathcal{P}_1(\widehat{t}nc). \end{aligned}$$

We have an analogous statement for $\mathcal{P}'$, hence our claim.

About (iii). We compute factor sets α_2 and α_2' for elements $\widehat{t}c$ and $\widehat{t}c'$ with $\widehat{t}$, $\widehat{t'} \in \mathcal{T}_2$ and $c, c' \in \mathrm{C}_{G_2}(N)$. This is sufficient since α_2 and α_2' are constant on M-cosets. Let $\widehat{t''} \in \mathcal{T}_2$, $m'' \in M$ and $c'' \in \mathrm{C}_{G_2}(N)$ such that $(\widehat{t}c)(\widehat{t'}c') = \widehat{t''}m''c''$ and let $\widetilde{c} \in \mathrm{C}_{G_1}(N)$ such that $tt' = t''m''\widetilde{c}$. Then $\alpha_2(\widehat{t}c, \widehat{t'}c') = \mu(\widetilde{c})\alpha_1(t'', \widetilde{c})\alpha_1(t, t')\widehat{\mu}(c)\widehat{\mu}(c')\widehat{\mu}(c'')^{-1}$, and α_2' can be expressed analogously. One deduces easily that α_2 coincides with α_2' on $H_2 \times H_2$. Therefore the pair $(\mathcal{P}_2, \mathcal{P}_2')$ gives the sought relation $(G_2, N, \theta) \geq_c (H_2, M, \varphi)$. □

Lemma 2.17 (Going to quotients, II)**.** *Suppose the relation $(G, N, \theta) \geq_c (H, M, \varphi)$ is given by the pair $(\mathcal{P}, \mathcal{P}')$. Let $Z \leq \ker\theta \cap \ker\varphi$ with $Z \lhd G$ and $\mathrm{C}_{G/Z}(N/Z) = \mathrm{C}_G(N)/Z$. Write $\overline{N} = N/Z$, $\overline{G} = G/Z$, $\overline{H} = H/Z$ and $\overline{M} = M/Z$. Denote by $\overline{\theta} \in \mathrm{Irr}(\overline{N})$, $\overline{\varphi} \in \mathrm{Irr}(\overline{M})$ the characters that lift to θ and φ. Then*

$$(\overline{G}, \overline{N}, \overline{\theta}) \geq_c (\overline{H}, \overline{M}, \overline{\varphi}).$$

Proof. Note first that $\mathcal{P}$ and $\mathcal{P}'$ are constant on Z-cosets. Hence they induce projective representations $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ of $\overline{G}$ and $\overline{H}$ associated with $\overline{\theta}$ and $\overline{\varphi}$. They satisfy the required properties since $\mathrm{C}_{G/Z}(N/Z) = \mathrm{C}_G(N)Z/Z$. □

Proposition 2.18 (Direct products)**.** *Suppose $(G_i, N_i, \theta_i) \geq_c (H_i, M_i, \varphi_i)$ for $i = 1, 2$. Then*

$$(G_1 \times G_2, N_1 \times N_2, \theta_1 \times \theta_2) \geq_c (H_1 \times H_2, M_1 \times M_2, \varphi_1 \times \varphi_2).$$

Proof. The group theoretical conditions 2.8(a) and (c) hold since $\mathrm{C}_{G_1 \times G_2}(N_1 \times N_2) = \mathrm{C}_{G_1}(N_1) \times \mathrm{C}_{G_2}(N_2)$.

Let $(\mathcal{P}_i, \mathcal{P}_i')$ be the projective representations associated with $(G_i, N_i, \theta_i) \geq_c (H_i, M_i, \varphi_i)$ for $i = 1, 2$. Let $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}'$ be defined as $\widetilde{\mathcal{P}}(x, y) = \mathcal{P}_1(x) \otimes \mathcal{P}_2(y)$ for $x \in G_1$, $y \in G_2$, respectively $\widetilde{\mathcal{P}}'(x, y) = \mathcal{P}_1'(x) \otimes \mathcal{P}_2'(y)$ for $x \in H_1$, $y \in H_2$. It is fairly clear that $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}'$ are projective representations associated with $\theta_1 \times \theta_2$ and $\varphi_1 \times \varphi_2$ respectively.

The factor set of $\widetilde{\mathcal{P}}$ satisfies $\widetilde{\alpha}\big((x_1, y_1), (x_2, y_2)\big) = \alpha_1(x_1, y_1)\alpha_2(x_2, y_2)$ for $x_i, y_i \in G_i$, and analogously for $\widetilde{\alpha}'$. We then get the conditions 2.8(b) and (d) from the corresponding statements for $(\mathcal{P}_i, \mathcal{P}_i')$ $i = 1, 2$. □

2.C. Wreath products. In this section we construct in Theorem 2.21 a new character triple, where the groups come from a wreath product construction. This finds an application in the "end game" for most reduction theorems. Often one can reduce to the situation that in an arbitrary finite group X the Fitting subgroup is central. This allows to find a normal subgroup $N \lhd X$ such that $N/(\mathrm{Z}(X) \cap N) \cong S \times \cdots \times S$ is a direct product of n copies of a simple group S. The action of X on N can be studied via the automorphism group of $N/(\mathrm{Z}(X) \cap N)$. Since the Fitting subgroup is central we can assume that S is non-abelian and therefore $\mathrm{Aut}(S \times \cdots \times S)$ is the wreath product $\mathrm{Aut}(S) \wr \mathrm{Sym}(n)$. So it is important to study the behavior of the above notions with regard to wreath products. We recall the main definitions.

Definition 2.19. Let n be a positive integer and G a finite group. Then the notation $G \wr \mathrm{Sym}(n)$ refers to the semi-direct product $G^n \rtimes \mathrm{Sym}(n)$ where $\mathrm{Sym}(n)$ acts on G^n by permuting the factors. Note that we are here using the convention that $\sigma\sigma'(i)$ coincides with $\sigma(\sigma'(i))$ for every $1 \leq i \leq n$. Namely the elements of $G \wr \mathrm{Sym}(n)$ are of the form $(g_1, \ldots, g_n)\sigma$ with $g_i \in G$ and $\sigma \in \mathrm{Sym}(n)$, and multiplication is defined by

$$(g_1, \ldots, g_n)\sigma(g_1', \ldots, g_n')\sigma' = (g_1 g'_{\sigma^{-1}(1)}, \ldots, g_n g'_{\sigma^{-1}(n)})\sigma\sigma'.$$

In the following we sketch how representations of the wreath product are constructed via tensor induction, see also [B, 3.15].

Definition 2.20. Let V be a $\mathbb{C}$-vector space and n a positive integer. One forms the tensor product space $W = V \otimes \cdots \otimes V$ (n terms). It is a left $\mathrm{Sym}(n)$-module via $\sigma(v_1 \otimes \cdots \otimes v_n) = (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$. One denotes the associated representation

$$\mathcal{R} : \mathrm{Sym}(n) \longrightarrow \mathrm{GL}(W).$$

For the elements $w \in W$ of the form $v_1 \otimes \cdots \otimes v_n$ ($v_i \in V$) one can check that every $\sigma \in \mathrm{Sym}(n)$ satisfies

$$\mathcal{R}(\sigma)(A_1 \otimes \cdots \otimes A_n)\mathcal{R}(\sigma)^{-1}w = A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(n)}w \text{ for every } A_1, \ldots, A_n \in \mathrm{GL}(V).$$

Since elements of the form $v_1 \otimes \cdots \otimes v_n$ form a generating set of W, we can conclude that the endomorphisms of W given by $\mathcal{R}(\sigma)(A_1 \otimes \cdots \otimes A_n)\mathcal{R}(\sigma)^{-1}$ and $A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(n)}$ coincide.

This leads to new character triple relations from old ones.

Theorem 2.21 (Wreath products)**.** *Assume a relation $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$ with non-trivial N. Let n be a positive integer. Denote $\widetilde{G} = G^n$, $\widetilde{N} = N^n$, $\widetilde{H} = H^n$, $\widetilde{M} = M^n$, and $\widetilde{\theta} = \theta \times \cdots \times \theta \in \mathrm{Irr}(\widetilde{N})$, $\widetilde{\varphi} = \varphi \times \cdots \times \varphi \in \mathrm{Irr}(\widetilde{M})$. Then*

$$(G \wr \mathrm{Sym}(n), \widetilde{N}, \widetilde{\theta}) \geq_{\mathrm{c}} (H \wr \mathrm{Sym}(n), \widetilde{M}, \widetilde{\varphi}).$$

Proof. Let $(\mathcal{P}, \mathcal{P}')$ be a pair of projective representations associated with the relation $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$. Let $\mathcal{R} : \mathrm{Sym}(n) \longrightarrow \mathrm{GL}_{\theta(1)n}(\mathbb{C})$ and $\mathcal{R}' : \mathrm{Sym}(n) \longrightarrow \mathrm{GL}_{\varphi(1)n}(\mathbb{C})$ as in Definition 2.20. Let $\widetilde{\mathcal{P}} : \widetilde{G} \rtimes \mathrm{Sym}(n) \longrightarrow \mathrm{GL}_{\theta(1)^n}(\mathbb{C})$ be defined by $\widetilde{\mathcal{P}}\big((g_1, \ldots, g_n)\sigma\big) = \big(\mathcal{P}(g_1) \otimes \cdots \otimes \mathcal{P}(g_n)\big)\mathcal{R}(\sigma)$ and $\widetilde{\mathcal{P}}' : \widetilde{H} \rtimes \mathrm{Sym}(n) \longrightarrow \mathrm{GL}_{\varphi(1)^n}(\mathbb{C})$ analogously. Then one checks easily that $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}'$ are projective representations associated with $\widetilde{\theta}$ and $\widetilde{\varphi}$. The group theoretic requirements (a) and (c) from Remark 2.8 hold (note that $\mathrm{C}_{G \wr \mathrm{Sym}(n)}(\widetilde{N}) = \mathrm{C}_G(N) \times \cdots \times \mathrm{C}_G(N)$). This also yields easily 2.8(d) for $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{P}}'$.

The factor set $\widetilde{\alpha}$ for $\widetilde{\mathcal{P}}$ satisfies $\widetilde{\alpha}\big((x_1, \ldots, x_n)\sigma, (y_1, \ldots, y_n)\tau\big) = \prod_{i=1}^n \alpha(x_i, y_{\sigma^{-1}(i)})$ where α is the factor set of $\mathcal{P}$. A similar formula holds for the factor set of $\widetilde{\mathcal{P}}'$. One deduces easily the condition 2.7(b). $\square$

3. A reduction theorem for the McKay conjecture

We now use the relation $\geq_c$ introduced in the preceding section to express compactly and prove the reduction theorem for the McKay Conjecture given by Isaacs-Malle-Navarro [IMN07].

We first need some group theoretical analysis of finite groups through the use of generalized Fitting subgroups. One is successively proving properties of a possible minimal counterexample. For example a character correspondence around the so-called Dade-Glauberman-Nagao correspondence from 3.C allows to assume that the Fitting subgroup of such a minimal counterexample is central. For such a group with central Fitting subgroup one can find a normal subgroup that is a central extension of some direct product of simple groups.

3.A. The McKay Conjecture. For a finite group G and a prime p we denote by $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1)_p = 1\}$ and by $\mathrm{Irr}_{p'}(G \mid \nu) = \mathrm{Irr}_{p'}(G) \cap \mathrm{Irr}(G|\nu)$, where ν is an irreducible character of some normal subgroup of G. Recall that whenever a group A acts on the finite group G we write $\mathrm{Irr}_A(G)$ for the A-invariant characters of G.

Conjecture 3.1 (Relative McKay). *Let $N \lhd G$ be finite groups, $P \in \mathrm{Syl}_p(G)$ and $\nu \in \mathrm{Irr}_P(N)$. Then*
$$|\mathrm{Irr}_{p'}(G \mid \nu)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P)N \mid \nu)|.$$

John McKay conjectured first the statement with $N = \{1\}$ (and $p = 2$ in the original paper [MK72]).

For the reduction theorem one considers mainly the above relative version that is known for example for p-solvable groups (Okuyama-Wajima [OW80]). In this relative version a normal subgroup is present hence allowing applications in the context of inductive proofs.

Proposition 3.2. *Let $N \lhd G$ with $\mathrm{Z}(G) \leq N$ and $P \in \mathrm{Syl}_p(G)$, $\nu \in \mathrm{Irr}_P(N)$. Assume Conjecture 3.1 holds for every $Z_1 \lhd G_1$ with the property $Z_1 \leq \mathrm{Z}(G_1)$ such that G_1/Z_1 is not isomorphic to a subgroup of G/N. Then $|\mathrm{Irr}_{p'}(G \mid \nu)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P)N \mid \nu)|$.*

Proof. Let $T = G_\nu$. Without loss of generality we can assume that $p \nmid \nu(1)$. Since ν is P-invariant, $p \nmid |G/T|$ and hence Clifford correspondence (Theorem 1.2) induces a bijection between $\mathrm{Irr}_{p'}(G \mid \nu)$ and $\mathrm{Irr}_{p'}(T \mid \nu)$. Applying Corollary 1.13, there is a character correspondence between $\mathrm{Irr}(T \mid \nu)$ and $\mathrm{Irr}(T^* \mid \lambda^*)$ for a finite group T^*, and some $\lambda^* \in \mathrm{Irr}(Z^*)$, where $Z^* \leq \mathrm{Z}(T^*)$ and $T^*/Z^* \cong T/N$. By its construction it restricts to a bijection between $\mathrm{Irr}_{p'}(T \mid \nu)$ and $\mathrm{Irr}_{p'}(T^* \mid \lambda^*)$.

Recall that $|T/N| \leq |G/N|$. Hence, by assumption, Conjecture 3.1 holds for the groups $Z^* \lhd T^*$, i.e. $|\mathrm{Irr}_{p'}(T^* \mid \lambda^*)| = |\mathrm{Irr}_{p'}(\mathrm{N}_{T^*}(P^*) \mid \lambda^*)|$ for $P^* \in \mathrm{Syl}_p(T^*)$. Again by the construction in Corollary 1.13 we get a bijection between $\mathrm{Irr}_{p'}(\mathrm{N}_{T^*}(P^*) \mid \lambda^*)$ and $\mathrm{Irr}_{p'}(\mathrm{N}_T(P)N \mid \nu)$ as $\mathrm{N}_{T^*}(P^*)/Z^* \cong \mathrm{N}_T(P)N/N$.

Clearly $(\mathrm{N}_G(P)N)_\nu = \mathrm{N}_T(P)N$ and by Clifford correspondence
$$|\mathrm{Irr}_{p'}(\mathrm{N}_T(P)N \mid \nu)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P)N \mid \nu)|. \tag{3.1}$$
□

3.B. Facts from finite group theory. Let us recall briefly the theory of the generalized Fitting subgroup of a finite group G, see also [A, §31]. The Fitting subgroup $F(G) = \prod_\ell O_\ell(G)$ is the normal subgroup of G generated by the maximal normal ℓ-subgroups, where ℓ ranges over all primes. Let $E(G)$ be the subgroup generated by the *components* of G, that are the quasi-simple subnormal subgroups of G. According to [A, 31.7] the quotient $E(G)/\mathrm{Z}(E(G))$ is the direct product of non-abelian simple groups. The generalized Fitting subgroup $F^*(G)$ of G is defined by $F^*(G) = E(G)F(G)$.

Based on the inclusion $\mathrm{C}_G(F^*(G)) \leq F^*(G)$ from [A, 31.13] the group $F^*(G)$ gives us a non-central normal subgroup containing $\mathrm{Z}(G)$ whenever G is non-abelian. In the following we consider various cases distinguishing possible normal subgroups of $F^*(G)$.

Proposition 3.3. *Assume that in the situation of Proposition 3.2, the group G satisfies $\mathrm{O}_p(G) \not\leq \mathrm{Z}(G)$. Then the McKay Conjecture 3.1 holds for G and p.*

Proof. For $N = \mathrm{O}_p(G)\mathrm{Z}(G)$ Proposition 3.2 proves $|\operatorname{Irr}_{p'}(G \mid \nu)| = |\operatorname{Irr}_{p'}(\mathrm{N}_G(P)N \mid \nu)|$. We also see that $\mathrm{N}_G(P)N = \mathrm{N}_G(P)\mathrm{O}_p(G) = \mathrm{N}_G(P)$. This implies $|\operatorname{Irr}_{p'}(G \mid \nu)| = |\operatorname{Irr}_{p'}(\mathrm{N}_G(P) \mid \nu)|$. □

3.C. The Glauberman correspondence. In order to prove that we can assume that a minimal counterexample has no non-central normal p'-subgroup we consider the following situation $K \lhd KP \lhd G$ where $K \lhd G$ is a p'-group and P is a p-subgroup of G such that $KP \lhd G$.

Theorem 3.4 (Glauberman correspondence)**.** *Let P be a p-group acting on a p'-group K. Then there exists a natural bijection $\operatorname{Irr}_P(K) \longrightarrow \operatorname{Irr}(\mathrm{C}_K(P))$ with $\theta \mapsto \theta^*$ such that θ^* is the only irreducible constituent of $\theta_{\mathrm{C}_K(P)}$ with p'-multiplicity.*

Proof. See [I, §13]. □

Corollary 3.5 (Dade, Puig, Navarro-Späth)**.** *Let P be a p-group acting on a p'-group K. Suppose $L = KP \lhd G$ with $K \lhd G$. There exists an $\mathrm{N}_G(P)$-equivariant bijection*

$$\Lambda : \operatorname{Irr}_{p'}(L) \longrightarrow \operatorname{Irr}_{p'}(\mathrm{N}_L(P))$$

such that

- $\Lambda(\operatorname{Irr}_{p'}(L \mid \theta)) = \operatorname{Irr}_{p'}(\mathrm{N}_L(P) \mid \theta^*)$ *for any* $\theta \in \operatorname{Irr}_P(K)$*, and*
- $(G_\chi, L, \chi) \geq_c (\mathrm{N}_G(P)_\chi, \mathrm{N}_L(P), \Lambda(\chi))$ *for any* $\chi \in \operatorname{Irr}_{p'}(L)$*.*

Proof. Puig has proven that θ extends to G if and only if θ^* extends to $\mathrm{N}_G(P)$. In [NS14, 5.13] this was used to construct a bijection with the above properties. □

Proposition 3.6. *Let $Z \leq \mathrm{Z}(G)$, $P \in \operatorname{Syl}_p(G)$ and $\nu \in \operatorname{Irr}(Z)$. Suppose, for some $L \lhd G$ and $P_0 := P \cap L$, that there exists an $\mathrm{N}_G(P_0)$-equivariant bijection*

$$\Lambda : \operatorname{Irr}_{p'}(L \mid \nu_{L\cap Z}) \longrightarrow \operatorname{Irr}_{p'}(\mathrm{N}_L(P_0) \mid \nu_{L\cap Z})$$

with $(G_\chi, L, \chi) \geq_c (\mathrm{N}_G(P_0)_\chi, \mathrm{N}_L(P_0), \Lambda(\chi))$ for any $\chi \in \operatorname{Irr}_{p'}(L)$. Then $|\operatorname{Irr}_{p'}(G \mid \nu)| = |\operatorname{Irr}_{p'}(\mathrm{N}_G(P_0) \mid \nu)|$.

$$\begin{tikzcd}
 & & G \arrow[dddd, no head, shift left=0, draw=none] \arrow[dd, no head] \\
\chi \arrow[r, dotted, no head] \arrow[dd, mapsto, "\Lambda"'] & L \arrow[ur, no head, "\lhd"] \arrow[dd, no head] & \\
 & & \mathrm{N}_G(P_0) \\
\Lambda(\chi) \arrow[r, dotted, no head] & \mathrm{N}_L(P_0) \arrow[ur, no head, "\lhd"'] &
\end{tikzcd}$$

Proof. Let $\nu_0 := \nu_{L\cap Z}$. Take a G-transversal $\mathcal{T}$ in $\{\chi \in \operatorname{Irr}_{p'}(L \mid \nu_0) \mid p \nmid |G : G_\chi|\}$. By the Sylow theorems $G = L\mathrm{N}_G(P_0)$, hence $\Lambda(\mathcal{T})$ is a $\mathrm{N}_G(P_0)$-transversal in

$$\{\eta \in \operatorname{Irr}_{p'}(\mathrm{N}_L(P_0) \mid \nu_0) \mid p \nmid |\mathrm{N}_G(P_0) : \mathrm{N}_G(P_0)_\eta|\}.$$

For $\chi \in \operatorname{Irr}(L \mid \nu_0)$ we denote by $\chi.\nu$ the unique element of $\operatorname{Irr}(LZ \mid \nu) \cap \operatorname{Irr}(LZ \mid \chi)$. Analogously for $\eta \in \operatorname{Irr}(\mathrm{N}_L(P_0) \mid \nu_0)$, $\{\eta.\nu\} := \operatorname{Irr}(\mathrm{N}_L(P_0)Z \mid \nu_0) \cap \operatorname{Irr}(\mathrm{N}_L(P_0)Z \mid \eta)$.

Now one gets the following equalities

$$(*)\qquad \mathrm{Irr}_{p'}(G \mid \nu) = \dot{\bigcup}_{\chi\in\mathcal{T}} \mathrm{Irr}_{p'}(G \mid \chi.\nu)$$

and

$$(**)\qquad \mathrm{Irr}_{p'}(\mathrm{N}_G(P_0) \mid \nu) = \dot{\bigcup}_{\eta\in\Lambda(\mathcal{T})} \mathrm{Irr}_{p'}(\mathrm{N}_G(P_0) \mid \eta.\nu).$$

Let $\chi \in \mathcal{T}$. Then there exist bijections $\sigma_{G_\chi} : \mathrm{Char}(G_\chi \mid \chi) \longrightarrow \mathrm{Char}(\mathrm{N}_G(P_0)_\chi \mid \Lambda(\chi))$ as in Theorem 2.2. They satisfy $\sigma_{G_\chi}(\mathrm{Irr}_{p'}(G_\chi \mid \chi.\nu)) = \mathrm{Irr}_{p'}(\mathrm{N}_G(P_0)_\chi \mid \Lambda(\chi).\nu)$.

Now using (*) and (**) above along with Clifford correspondence from Theorem 1.2, one computes

$$\begin{aligned}
|\,\mathrm{Irr}_{p'}(G \mid \nu)| &= \sum_{\chi\in\mathcal{T}} |\,\mathrm{Irr}_{p'}(G_\chi \mid \chi.\nu)| \\
&= \sum_{\eta\in\Lambda(\mathcal{T})} |\,\mathrm{Irr}_{p'}(\mathrm{N}_G(P_0)_\eta \mid \eta.\nu)| \\
&= |\,\mathrm{Irr}_{p'}(\mathrm{N}_G(P_0) \mid \nu)|.
\end{aligned}$$

□

For the later applications we give a generalisation of the above result, that follows from the same arguments.

Corollary 3.7. *Let $Z \le \mathrm{Z}(G)$, $P \in \mathrm{Syl}_p(G)$ and $\nu \in \mathrm{Irr}(Z)$. Suppose, for some $L \lhd G$ and $P_0 := P \cap L$, that there exists an $\mathrm{N}_G(P_0)$-invariant subgroup M of L and an $\mathrm{N}_G(P_0)$-equivariant bijection*

$$\Lambda : \mathrm{Irr}_{p'}(L \mid \nu_{L\cap Z}) \longrightarrow \mathrm{Irr}_{p'}(M \mid \nu_{L\cap Z})$$

with $(G_\chi, L, \chi) \ge_c (M\mathrm{N}_G(P_0)_\chi, M, \Lambda(\chi))$ for any $\chi \in \mathrm{Irr}_{p'}(L)$. Then $|\,\mathrm{Irr}_{p'}(G \mid \nu)| = |\,\mathrm{Irr}_{p'}(\mathrm{N}_G(P_0) \mid \nu)|$.

Corollary 3.8. *If in the situation of Proposition 3.2, the group G satisfies $\mathrm{O}_{p'}(G) \not\le \mathrm{Z}(G)$, then the McKay conjecture holds.*

Proof. For $K = \mathrm{O}_{p'}(G)$ and $L = KP$ we apply the corollaries 3.5 and Proposition 3.6 to obtain the statement. □

In a proof of Conjecture 3.1 by induction on the order of $|G|$, we can assume with Proposition 3.2 that $F(G)\mathrm{O}_{p'}(G) \le \mathrm{Z}(G)$ and that all components S of $E(G)/\mathrm{Z}(E(G))$ have order divisible by p. Let S be a non-abelian simple group occuring as a direct factor in $E(G)/\mathrm{Z}(E(G))$ and let $L/\mathrm{Z}(E(G))$ be the subgroup of $E(G)/\mathrm{Z}(E(G))$ that is the product of the components isomorphic to S. Then $L \lhd G$ and $L/\mathrm{Z}(E(G)) \cong S \times \cdots \times S$.

Definition 3.9 (Inductive McKay condition - "McKay-good")**.** Let S be a non-abelian simple group with $p \mid |S|$ and X the universal covering group of S (i.e., $[X, X] = X$, $X/\mathrm{Z}(X) \cong S$ and $\mathrm{Z}(X) \cong H^2(S, \mathbb{C}^\times)$).

We say that S is *McKay-good for p* if and only if for some $Q \in \mathrm{Syl}_p(X)$ and $\Gamma := \mathrm{Aut}(X)_Q$, there exist:

(i) some Γ-invariant group M with $\mathrm{N}_X(Q) \le M \lneq X$,

(ii) some Γ-equivariant bijection $\Lambda : \mathrm{Irr}_{p'}(X) \longrightarrow \mathrm{Irr}_{p'}(M)$ such that

(iii) $(X \rtimes \Gamma_\chi, X, \chi) \ge_c (M \rtimes \Gamma_\chi, M, \Lambda(\chi))$ for every $\chi \in \mathrm{Irr}_{p'}(X)$.

Note that in the above, the group M in (i) is allowed to be different from $\mathrm{N}_X(Q)$ and possibly better behaved group theoretically, see also Example 3.11. In (iii), the fact that $M \rtimes \Gamma_\chi$ actually stabilizes $\Lambda(\chi)$ comes from the fact that Λ is Γ-equivariant.

Remark 3.10. (a) Let G be a finite group with normal subgroup L. If $[L, L]$ and X are isomorphic, the above condition ensures that a character correspondence analogous to the one in Corollary 3.5 exists, where P is then a Sylow p-subgroup of L.

(b) The verification of (iii) is simplified by the Butterfly Theorem 2.16: it suffices to find for every $\chi \in \mathrm{Irr}_{p'}(X)$ a group O such that $X \lhd O$, $O/\mathrm{C}_O(X) \cong \mathrm{Aut}(X)_\chi$ (via the natural morphism) and $(O, X, \chi) \geq_{\mathrm{c}} (M\mathrm{N}_O(Q)_\chi, M, \Lambda(\chi))$. Note that in this context O can be chosen depending on χ.

(c) It is sufficient to prove (iii) for some Γ-transversal in $\mathrm{Irr}_{p'}(X)$, since $\geq_{\mathrm{c}}$ is compatible with conjugation.

In order to get an idea why the introduction of this group M in Definition 3.9 is useful we give an example where M is chosen in a way that it is sometimes different from the normalizer of a Sylow p-group in X. This choice has proven to be handier in the verification.

Example 3.11. Let S be the simple group $\mathrm{PSL}_n(q)$ for q a prime power and $n \geq 2$. Assume the prime p divides $q-1$ and $p \geq 5$, and $G := \mathrm{SL}_n(q)$ is its universal covering, hence the Schur multiplier is non-exceptional.

Let $T \leq M = \mathrm{N}_G(T) \leq G$ the subgroup of diagonal and monomial matrices. It is clear from the order of G that M contains a Sylow p-subgroup P of G, whenever $p > n$. One can then choose P to be the Sylow p-subgroup of T. Then $\mathrm{N}_G(P) = M$, see [Ma07, Thm. 5.14].

When $p \leq n$, then one can show that the Sylow p-subgroup P_0 of T is a characteristic subgroup of P with $\mathrm{N}_G(P_0) = M$ and therefore M satisfies the condition (i) of Definition 3.9. The group M does not depend on the the specific value of p any more, as long as $p \mid (q-1)$. Moreover the representations of M are easier to study than the ones of $\mathrm{N}_G(P)$.

Theorem 3.12. *Assume that S is a non-abelian simple group that is McKay-good for p with X, Q, Γ and M as in Definition 3.9. Let $m \geq 1$ be an integer, take $\widetilde{X} = X^m$, $\widetilde{Q} = Q^m \in \mathrm{Syl}_p(\widetilde{X})$ and $\widetilde{\Gamma} = \Gamma \wr \mathrm{Sym}(m) = \Gamma^m \rtimes \mathrm{Sym}(m) = \mathrm{Aut}(\widetilde{X})_{\widetilde{Q}}$. Then:*

(a) $\widetilde{M} := M^m$ is $\widetilde{\Gamma}$-stable

(b) there exists some $\widetilde{\Gamma}$-equivariant bijection $\widetilde{\Lambda} : \mathrm{Irr}_{p'}(\widetilde{X}) \longrightarrow \mathrm{Irr}_{p'}(\widetilde{M})$, and

(c) we have $(\widetilde{X} \rtimes \widetilde{\Gamma}_\varphi, \widetilde{X}, \varphi) \geq_{\mathrm{c}} (\widetilde{M} \rtimes \widetilde{\Gamma}_\varphi, \widetilde{M}, \widetilde{\Lambda}(\varphi))$ for every $\varphi \in \mathrm{Irr}_{p'}(\widetilde{X})$.

Proof. By definition $\widetilde{M}$ is $\widetilde{\Gamma}^m$- and $\mathrm{Sym}(m)$-stable, this proves part (a).

We define $\widetilde{\Lambda}$ by $\widetilde{\Lambda}(\psi) = \Lambda(\chi_1) \times \cdots \times \Lambda(\chi_m)$ for every $\psi \in \mathrm{Irr}_{p'}(X^m)$ written as $\chi_1 \times \cdots \times \chi_m$ for $\chi_i \in \mathrm{Irr}_{p'}(X)$. It is clear that $\widetilde{\Lambda}$ is bijective and $\widetilde{\Gamma}$-equivariant. This is part (b).

We now check (c). Assume $\psi = \chi \times \cdots \times \chi$ (m times) for some $\chi \in \mathrm{Irr}_{p'}(X)$. Then $\widetilde{\Gamma}_\psi = \Gamma_\chi \wr \mathrm{Sym}(m)$ and we must show $((X \rtimes \Gamma_\chi) \wr \mathrm{Sym}(m), \widetilde{X}, \psi) \geq_{\mathrm{c}} ((M \rtimes \Gamma_\chi) \wr \mathrm{Sym}(m), \widetilde{M}, \widetilde{\Lambda}(\psi))$ which holds by Theorem 2.21.

Assume $\psi = \chi_1 \times \cdots \times \chi_m$ such that for every $1 \leq i < j \leq n$, the characters χ_i and χ_j are always either equal or not Γ-conjugate. Then $\widetilde{\Gamma}_\chi$ is a direct product of groups of the form $\Gamma_{\chi_i} \wr \mathrm{Sym}(m_i)$. The statement in (c) for ψ then follows from Proposition 2.18 and Theorem 2.21.

Now observe that every character $\varphi \in \mathrm{Irr}_{p'}(\widetilde{X})$ is $\widetilde{\Gamma}$-conjugate to one considered before, i.e., that there exists some ψ as considered before and some $h \in \widetilde{\Gamma}$ such that $\varphi = \psi^h$. Now by straightforward computations one shows that $(\widetilde{X} \rtimes \widetilde{\Gamma}_\psi, \widetilde{X}, \psi) \geq_{\mathrm{c}} (\widetilde{M} \rtimes \widetilde{\Gamma}_\psi, \widetilde{M}, \widetilde{\Lambda}(\psi))$ implies $(\widetilde{X} \rtimes \widetilde{\Gamma}_\psi^h, \widetilde{X}, \psi^h) \geq_{\mathrm{c}} (\widetilde{M} \rtimes \widetilde{\Gamma}_\psi^h, \widetilde{M}, \widetilde{\Lambda}(\psi)^h)$. This can be rewritten as

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\varphi, \widetilde{X}, \varphi) \geq_{\mathrm{c}} (\widetilde{M} \rtimes \widetilde{\Gamma}_\varphi, \widetilde{M}, \widetilde{\Lambda}(\varphi)).$$

$\square$

Remark 3.13 (Universal covering groups). Let (S, X) be as before and L any group with $L = [L, L]$ and $L/\mathrm{Z}(L) \cong S^m$. It is well-known that then there exists an epimorphism $\epsilon : X^m \to L$ with kernel in $\mathrm{Z}(X^m)$.

Corollary 3.14. *Assume that S is simple, non-abelian and McKay-good for p in the sense of Definition 3.9. Let L and G be groups such that $\mathrm{Z}(G) \leq L \lhd G$ and $L/\mathrm{Z}(L) \cong S^m$. Let $P \in \mathrm{Syl}_p(G)$ and $P_1 := P \cap L$. Then*

$$|\operatorname{Irr}_{p'}(G \mid \nu)| = |\operatorname{Irr}_{p'}(M\mathrm{N}_G(P_1) \mid \nu)|$$

for every $\nu \in \operatorname{Irr}(\mathrm{Z}(G))$ and some $\mathrm{N}_G(P_0)$-stable $M \lneq L$ with $\mathrm{N}_L(P_0) \leq M$.

Proof. Let $L_0 = [L, L]$ and $\epsilon : X^m = \widetilde{X} \longrightarrow L_0$ be an epimorphism as in Remark 3.13. For $Z := \ker(\epsilon)$, $P_0 := L_0 \cap P_1$ and $\Upsilon := \operatorname{Aut}(\widetilde{X})_{\widetilde{Q},Z}$ we apply Theorem 3.12 and get some Υ-stable subgroup $\widetilde{M} \leq \widetilde{X}$ with $\widetilde{M}/Z \geq \mathrm{N}_{L_0}(P_0)$ and a Υ-equivariant bijection

$$\Omega : \operatorname{Irr}_{p'}(\widetilde{X} \mid 1_Z) \longrightarrow \operatorname{Irr}_{p'}(\widetilde{M} \mid 1_Z)$$

with certain properties. The property 3.12(c) implies $(\widetilde{X} \rtimes \Upsilon_\psi, \widetilde{X}, \psi) \geq_{\mathrm{c}} (\widetilde{M} \rtimes \Upsilon_\psi, \widetilde{M}, \Omega(\psi))$ for every $\psi \in \operatorname{Irr}_{p'}(\widetilde{X} \mid 1_Z)$ since $\widetilde{\Gamma}_\psi \geq \Upsilon_\psi$.

$$\begin{tikzcd}
G \arrow[dr, dash] & & & & \\
& L \arrow[dr, dash] \arrow[ddd, dash, "S^m\cong"'] & & & \\
& & L_0 \arrow[d, dash] & \widetilde{X} \arrow[l, "\epsilon"'] \arrow[d, dash] & \operatorname{Irr}_{p'}(\widetilde{X} \mid 1_Z) \arrow[d, "\Omega"'] \\
& & M \arrow[dd, dash] & \widetilde{M} \arrow[l, "\epsilon_M"'] & \operatorname{Irr}_{p'}(\widetilde{M} \mid 1_Z) \\
& \mathrm{Z}(G) \arrow[dr, dash] & & & \\
& & \mathrm{Z}(G) \cap L_0 & &
\end{tikzcd}$$

Note that $L_0 \cong X^m/Z$ via ϵ. We apply Lemma 2.17 to the above, where we use that $\operatorname{Aut}(\widetilde{X}) = \operatorname{Aut}(\widetilde{S}) = \operatorname{Aut}(S) \rtimes \operatorname{Sym}(m)$ by [A, Ex 33.6] and hence $\mathrm{C}_{\widetilde{X} \rtimes \Upsilon_\psi}(\widetilde{X})/Z = \mathrm{C}_{\widetilde{X}/Z \rtimes \Upsilon_\psi}(\widetilde{X}/Z)$. This implies

$$(\widetilde{X}/Z \rtimes \Upsilon_\psi, \widetilde{X}/Z, \overline{\psi}) \geq_{\mathrm{c}} (\widetilde{M}/Z \rtimes \Upsilon_\psi, \widetilde{M}/Z, \overline{\Omega(\psi)})$$

for every $\psi \in \operatorname{Irr}_{p'}(\widetilde{X} \mid 1_Z)$, where $\overline{\psi}$ is the character of $\widetilde{X}/Z$ induced by ψ and $\overline{\Omega(\psi)}$ the character of $\widetilde{M}/Z$ induced by $\Omega(\psi)$. Let $M := \epsilon(\widetilde{M})$. Then Ω induces a bijection

$$\overline{\Omega} : \operatorname{Irr}_{p'}(L_0) \longrightarrow \operatorname{Irr}_{p'}(M).$$

Using $\operatorname{Aut}(L_0) = \operatorname{Aut}(\widetilde{X})_Z$, Theorem 2.16 implies that by the above every character $\chi \in \operatorname{Irr}_{p'}(L_0)$ satisfies

$$(G_\chi, L_0, \chi) \geq_{\mathrm{c}} (M\mathrm{N}_{G_\chi}(P_0), M, \overline{\Omega}(\chi)).$$

The arguments from Corollary 3.7 now imply

$$|\operatorname{Irr}_{p'}(G \mid \nu)| = |\operatorname{Irr}_{p'}(M\mathrm{N}_G(P_0) \mid \nu)|$$

for every $\nu \in \operatorname{Irr}(\mathrm{Z}(G))$. Since $L = L_0\,\mathrm{Z}(G)$ we see that $P_0\,\mathrm{Z}(G) \geq P_1$ and hence $\mathrm{N}_G(P_0) = \mathrm{N}_G(P_1)$. Accordingly the above proves the stated equality. $\square$

3.D. The reduction theorem of Isaacs-Malle-Navarro. We present here the reduction theorem for the McKay conjecture due to Isaacs-Malle-Navarro. We apply for its proof the language introduced so far.

Recall that a group S is said to be *involved* in the group G whenever $S \cong G_1/G_2$ for some $G_2 \lhd G_1 \leq G$.

Theorem 3.15 (Isaacs-Malle-Navarro [IMN07])**.** *Let G be a finite group and assume that every simple non-abelian group S with $p \mid |S|$ involved in G is McKay-good for p. Then the McKay Conjecture 3.1 holds for G and p.*

Proof by induction on $|G/\mathrm{Z}(G)|$. Assume Conjecture 3.1 holds for every $G_1 \rhd \mathrm{Z}(G_1)$ that satisfies $|G_1/\mathrm{Z}(G_1)| \lneq |G/\mathrm{Z}(G)|$ and for which $G_1/\mathrm{Z}(G_1)$ is involved in G.

Without loss of generality we may assume that G is non-abelian, $\mathrm{O}_p(G) \leq \mathrm{Z}(G)$ (Proposition 3.3) and $\mathrm{O}_{p'}(G) \leq \mathrm{Z}(G)$ (Corollary 3.8). Then there exists some $L \lhd G$ such that $L/\mathrm{Z}(G) \cong S^m$ for some non-abelian simple group S with $p \mid |S|$ and $m \geq 1$. Since S is McKay-good, Corollary 3.13 implies that for some $P_0 \in \mathrm{Syl}_p(L)$, some $\mathrm{N}_G(P_0)$-stable $M \lneq [L, L]$, , and any $\nu \in \mathrm{Irr}(\mathrm{Z}(G))$ one has

$$|\mathrm{Irr}_{p'}(G \mid \nu)| = |\mathrm{Irr}_{p'}(M\mathrm{N}_G(P_0) \mid \nu)|.$$

Because of $M\mathrm{N}_G(P_0) \lneq G$, the inductive hypothesis gives

$$|\mathrm{Irr}_{p'}(M\mathrm{N}_G(P_0) \mid \nu)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P) \mid \nu)|$$

for some $P \in \mathrm{Syl}_p(G)$ with $P_0 \lhd P$. Combining with the equality above, we get indeed $|\mathrm{Irr}_{p'}(G \mid \nu)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P) \mid \nu)|$. This completes our proof. □

3.E. The case of a cyclic outer automorphism group. We now give a hint on how to verify the inductive McKay condition from Definition 3.9. According to the classification of finite simple groups, see [A, §47], one has to consider alternating groups, finite simple groups of Lie type and the sporadic simple groups.

We show next how the checking of the inductive McKay condition simplifies when the simple group S has cyclic $\mathrm{Aut}(S)/S$. This is the case for the sporadic groups, most alternating groups but also groups of Lie type whose root system has trivial fundamental group (types E_8, F_4, G_2 and their twisted versions), see [GLS, §2.5, §5.3].

Lemma 3.16. *Let (G, N, θ) and (H, M, φ) be two character triples with* **cyclic** *G/N such that $G = NH$, $M = H \cap N$ and $\mathrm{C}_G(N) \leq H$. (These are the group theoretic requirements for the relation $\geq_{\mathrm{c}}$ from Definition 2.7). Assume that $\theta_{\mathrm{Z}(N)}$ and $\varphi_{\mathrm{Z}(N)}$ are multiples of the same element of $\mathrm{Irr}(\mathrm{Z}(N))$. Then*

$$(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi).$$

Proof. Since G/N is cyclic, θ extends to G. Because of $G = NH$ and $M = H \cap N$ the quotient H/M is cyclic and hence φ extends to H. Let $\widetilde{\theta}$ and $\widetilde{\varphi}$ be some extensions obtained by that. By Lemma 2.15 we have to ensure that $\mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(N)}) = \mathrm{Irr}(\widetilde{\varphi}_{\mathrm{C}_G(N)})$. If $\mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(N)}) \neq \mathrm{Irr}(\widetilde{\varphi}_{\mathrm{C}_G(N)})$ there exists some character $\kappa \in \mathrm{Irr}(\mathrm{C}_G(N))$ such that $\mu\kappa = \mu'$ where $\{\mu\} = \mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(N)})$ and $\{\mu'\} = \mathrm{Irr}(\widetilde{\varphi}_{\mathrm{C}_G(N)})$. By definition the character κ is the lift of a character of $\mathrm{C}_G(N)/\mathrm{Z}(N)$. The character induces one of $\mathrm{C}_G(N)N/N$. Since G/N is cyclic, κ as character of $\mathrm{C}_G(N)N/N$ extends to some $\widetilde{\kappa} \in \mathrm{Irr}(G/N)$. The character $\widetilde{\theta}\widetilde{\kappa}$ is an extension of θ to G. By the definition of $\widetilde{\kappa}$ the characters $\widetilde{\theta}\widetilde{\kappa}$ and $\widetilde{\varphi}$ satisfy $\mathrm{Irr}((\widetilde{\theta}\widetilde{\kappa})_{\mathrm{C}_G(N)}) = \mathrm{Irr}(\widetilde{\varphi}_{\mathrm{C}_G(N)})$. This proves the statement. □

Proposition 3.17. *Let S be a simple non-abelian group with $p \mid |S|$ and $\mathrm{Aut}(S)/S$ is cyclic. Assume that for the universal covering group X of S the conditions (i) and (ii) of the inductive McKay condition of Definition 3.9 are satisfied and that $\Lambda(\mathrm{Irr}_{p'}(X \mid \nu)) = \mathrm{Irr}_{p'}(M \mid \nu)$ for any $\nu \in \mathrm{Irr}(\mathrm{Z}(X))$. Then S is McKay-good for p.*

Proof. We just have to check condition 3.9(iii). Since $\mathrm{Aut}(X)/S \cong \mathrm{Aut}(S)/S$ by [A, Ex 33.6] one may build an overgroup $X \lhd Y$ such that Y induces $\mathrm{Aut}(X)$ on X and Y/X is cyclic. Let $\chi \in \mathrm{Irr}_{p'}(X)$ and $Q \in \mathrm{Syl}_p(X)$. By Lemma 3.16 above, we have

$$(Y_\chi, X, \chi) \geq_{\mathrm{c}} (M\mathrm{N}_Y(Q)_\chi, M, \Lambda(\chi)).$$

Now the Butterfly Theorem 2.16 gives our claim

$$(X \rtimes \mathrm{Aut}(X)_{Q,\chi}, X, \chi) \geq_{\mathrm{c}} (M \rtimes \mathrm{Aut}(X)_{Q,\chi}, M, \Lambda(\chi)).\square$$

4. Other reduction theorems

In the preceding section we have explained in length the use of character triples in our rephrasement of the reduction theorem for the McKay Conjecture. Recently, character triples have been mostly used in a context involving blocks in relation with Brauer's height zero conjecture and Alperin-McKay's conjecture, see [NS14]. In this more involved context, the notion of ordered character triples (relations $\geq$ and $\geq_{\mathrm{c}}$ of Definitions 2.1 and 2.7) made the main ideas more accessible.

In the following section, we just give an overview of the key ideas around blocks and modular characters. This leads to the definition of the relation $\geq_{\mathrm{b}}$, see Definition 4.2. The main application is the reduction theorem for Alperin's weight conjecture Theorem 4.22.

4.A. Blocks and the generalized Dade-Glauberman-Nagao correspondence. Our aim in this section is to extend the theory presented so far to incorporate blocks and heights of characters, with applications to Brauer characters.

Let us recall some notation. If A is a finite subset of a group, one denotes by A^+ the sum $\sum_{a\in A} a$ of elements of A in the group algebra. This will apply mainly to the additive group of group algebras. If G is a finite group and $g \in G$, $\mathrm{cc}_G(g)$ denotes the G-conjugacy class of g.

We keep p a prime. We let $(K, \mathcal{O}, k)$ be a p-modular system in the sense of [NT, §3.6]. This implies that $\mathcal{O}$ is a complete valuation ring with $\mathcal{O}/J(\mathcal{O}) = k$ of characteristic p, K is the fraction field of $\mathcal{O}$ and the algebra KG, respectively $kG/J(kG)$, is isomorphic to a product of matrix algebras over K, respectively k. The surjective map $\mathcal{O} \longrightarrow k$ is denoted by $\lambda \mapsto \lambda^*$.

For a finite group G, the p-blocks of G are the blocks of the group algebra kG, so that $kG = \oplus_{B\in \mathrm{Bl}(G)} B \cong \prod_{B\in\mathrm{Bl}(G)} B$ the latter an isomorphism of k-algebras. This clearly induces a partition of Brauer characters. We also have a similar decomposition $\mathcal{O}G = \oplus_{B\in\mathrm{Bl}(G)} \widehat{B} \cong \prod_{B\in\mathrm{Bl}(G)} \widehat{B}$ giving the above by the reduction map $\lambda \mapsto \lambda^*$. The latter induces in turn a partition $\mathrm{Irr}(G) = \cup_{B\in\mathrm{Bl}(G)} \mathrm{Irr}(B)$. If $B \subseteq kG$ is a sum of blocks, one writes $\mathrm{Irr}(B)$ for the union of sets $\mathrm{Irr}(b)$ where b is any block contained in B.

Notation 4.1. For $\theta \in \mathrm{Irr}(G)$ we define $\mathrm{bl}(\theta) \in \mathrm{Bl}(G)$ as the block satisfying $\theta \in \mathrm{Irr}(\mathrm{bl}(\theta))$. A p-block b of G induces a unique algebra morphism $\lambda_b : \mathrm{Z}(kG) \longrightarrow k$. Furthermore, λ_b satisfies $\lambda_b(\mathrm{cc}_G(x)^+) = \left(\frac{|\mathrm{cc}_G(x)|\chi(x)}{\chi(1)}\right)^*$ for any $\chi \in \mathrm{Irr}(b)$ and $x \in G$.

The defect groups of b are the minimal elements in

$$\{P \mid P \in \mathrm{Syl}_p(\mathrm{C}_G(x)),\ \lambda_b(\mathrm{cc}_G(x)^+) \neq 0\},$$

where x ranges over the elements of G, see [Na1, 4.3] or [NT, 5.1.11]. One uses the notation $\mathrm{Bl}(G \mid P)$ to denote the set of blocks of G having P as a defect group.

For groups $H \leq G$ and $b \in \mathrm{Bl}(H)$, one defines $\lambda_b^G : \mathrm{Z}(kG) \longrightarrow k$ by $\lambda_g^G(\mathrm{cc}_G(x)^+) = \lambda_b((\mathrm{cc}_G(x)\cap H)^+)$. If moreover λ_b^G is an algebra morphism then one denotes by $b^G \in \mathrm{Bl}(G)$ the corresponding block, known as the block of G induced by b. This block is defined when $H \geq \mathrm{C}_G(Q)$ where Q is a defect group of b in H, see [Na1, 4.14]. Brauer correspondence is the bijection

$$\mathrm{Bl}(\mathrm{N}_G(P) \mid P) \to \mathrm{Bl}(G \mid P) \text{ given by } b \mapsto b^G,$$

see [Na1, 4.17]. If $N \lhd G$, $B \in \mathrm{Bl}(G)$ and $b \in \mathrm{Bl}(N)$, we say that B covers b if and only if $\mathrm{Irr}(B) \cap \mathrm{Irr}(G \mid \varphi) \neq \emptyset$ for some $\varphi \in \mathrm{Irr}(B)$. Those blocks form the set $\mathrm{Bl}(G \mid b)$.

Definition 4.2. Let (G, N, θ) and (H, M, φ) be two character triples. Then we write

$$(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$$

whenever the following conditiions are satisfied

(i) $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$,

(ii) a defect group D of $\mathrm{bl}(\varphi)$ satisfies $\mathrm{C}_G(D) \leq H$, and

(iii) there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ such that the pair $(\mathcal{P}, \mathcal{P}')$ is associated with $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$ and the maps σ_J $(N \leq J \leq G)$ induced by $(\mathcal{P}, \mathcal{P}')$ as in Theorem 2.2 satisfy

$$\mathrm{bl}(\psi) = \mathrm{bl}(\sigma_J(\psi))^J$$

for every subgroup J with $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$.

Remark 4.3. (a) Because of (i) some defect group $\widetilde{D}$ of $\mathrm{bl}(\sigma_J(\psi))$ satisfies $D \leq \widetilde{D}$ and hence $\mathrm{C}_J(\widetilde{D}) \leq \mathrm{C}_J(D) \leq H \cap J$. In particular the block $\mathrm{bl}(\sigma_J(\psi))^J$ is defined.

(b) Let $B \in \mathrm{Bl}(G)$ and B' be the sum of blocks $c \in \mathrm{Bl}(H \mid \mathrm{bl}(\psi))$ with $c^G = B$. Then the relation $(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$ implies $|\mathrm{Irr}(B \mid \theta)| = |\mathrm{Irr}(B' \mid \psi)|$, where $\mathrm{Irr}(B \mid \theta) := \mathrm{Irr}(B) \cap \mathrm{Irr}(G \mid \theta)$.

(c) Assume that $\mathrm{bl}(\varphi)$ and $\mathrm{bl}(\theta)$ have a common defect group D and that $\mathrm{N}_N(D) = M$. Then B' (defined as in (b) above) is just one block, the Harris-Knörr correspondent from [Na1, 9.28] of B.

$$\begin{tikzcd}
 & & G & \mathrm{Bl}(G \mid b) & B = (B')^G \\
\mathrm{bl}(\theta) = b & N \arrow[ur, dash] \arrow[dd, dash] & & & \\
 & & \mathrm{N}_G(D) \arrow[uu, dash] & \mathrm{Bl}(\mathrm{N}_G(D) \mid b') \arrow[uu, "1-1"'] & B' \arrow[uu, mapsto] \\
\mathrm{bl}(\varphi) = b' & \mathrm{N}_N(D) \arrow[ur, dash] & & &
\end{tikzcd}$$

The following is an analogue of Lemma 2.15:

Proposition 4.4. *Let $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$ with $H \geq \mathrm{C}_G(D)$ for some defect group D of $\mathrm{bl}(\varphi)$. Assume θ is the restriction of some $\widetilde{\theta} \in \mathrm{Irr}(G)$. Then $(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$ if and only if φ is the restriction of some $\widetilde{\varphi} \in \mathrm{Irr}(H)$ such that*

(a) $\widetilde{\varphi}_{\mathrm{C}_G(N)}$ and $\widetilde{\theta}_{\mathrm{C}_G(N)}$ have the same irreducible constituents (with possibly different multiplicities),

(b) $\mathrm{bl}(\widetilde{\varphi}_{J \cap H})^J = \mathrm{bl}(\widetilde{\theta}_J)$ for every J with $N \leq J \leq G$.

$$\begin{tikzcd}
 & & & G \arrow[r, dotted, dash] \arrow[dl, dash] \arrow[ddd, dash] & \widetilde{\theta} \\
 & & J \arrow[dl, dash] & & \\
\theta \arrow[r, dotted, dash] & N \arrow[ddd, dash] & & & \\
 & & & H \arrow[r, dotted, dash] \arrow[dl, dash] & \widetilde{\varphi} \\
 & & J \cap H \arrow[r, dotted, dash] \arrow[dl, dash] & \mathrm{bl}(\widetilde{\varphi}_{J\cap H}) & \\
\varphi \arrow[r, dotted, dash] & M & & &
\end{tikzcd}$$

Proof. We first assume that $(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$ and that σ_G is given by the associated character triple isomorphism.

Let $\widetilde{\varphi} := \sigma_G(\widetilde{\theta})$. By Corollary 2.4, $\widetilde{\varphi}_M = \sigma_G(\widetilde{\theta})_M = \sigma_N(\widetilde{\theta}_N) = \varphi$. The character $\widetilde{\varphi}$ satisfies (a) by Remark 2.12. Part (b) follows directly from Definition 4.2(ii).

For the other direction assume that an extension $\widetilde{\varphi}$ with the described properties exists. Then for $\eta \in \mathrm{Irr}(J \mid 1_N)$, $\lambda_{\mathrm{bl}(\widetilde{\theta}_J\eta)}$ and $\lambda_{\mathrm{bl}(\widetilde{\varphi}_J\eta)}$ can be explicitly computed with $\lambda_{\mathrm{bl}(\widetilde{\theta}_J)}$ and $\lambda_{\mathrm{bl}(\overline{\eta})}$ for $\overline{\eta} \in \mathrm{Irr}(G/N)$.

Taking now $(\mathcal{P}, \mathcal{P}')$ as linear representations affording $\widetilde{\theta}$ and $\widetilde{\varphi}$, an easy explicit computation shows that the induced maps σ_J satisfy the condition of Definition 4.2(iii). $\square$

Remark 4.5. One can generalize Proposition 4.4 and prove that $(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$ is equivalent to the existence of projective representations $(\mathcal{P}, \mathcal{P}')$ associated with $(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$ with the property that $\mathcal{P}(\mathrm{cc}_J(x)^+)^*$ and $\mathcal{P}'((\mathrm{cc}_J(x) \cap H)^+)^*$ are scalar matrices to the same scalar for every $x \in G$ and $J = \langle N, x \rangle$.

4.B. An inductive condition for the Alperin-McKay conjecture. One proves the following analogue of Proposition 2.18 and Theorem 2.21.

Theorem 4.6. *The relation $\geq_{\mathrm{b}}$ is compatible with taking direct products and wreath products. An analogue of the Butterfly Theorem 2.16 also holds.*

Proof. A proof of this statement follows the arguments given previously. A guideline how to adapt the considerations to blocks can be found in [S17]. More precisely for the relation $\sim_N$ considered there coincides with $\geq_{\mathrm{b}}$ whenever the groups present in the considered character triples satisfy certain properties. Hence the claim follows from [S17, 5.1, 5.2 and 5.3]. $\square$

Note that on the other hand Lemma 2.17 (going to quotients) has no clear analogue since there is no bijection between $\mathrm{Bl}(G/Z)$ and $\mathrm{Bl}(G \mid \mathrm{bl}(1_Z))$ whenever Z is a normal p-group with $Z \not\leq \mathrm{Z}(G)$.

Let us now recall the notion of height of a character in a p-block. If $\chi \in \mathrm{Irr}(G)$, then $\chi(1)_p = |G : D|_p p^{\mathrm{ht}(\chi)}$ for some non-negative integer $\mathrm{ht}(\chi) \geq 0$ where D is a defect group of $\mathrm{bl}(\chi)$. This is called the *height* of χ.

When B is a p-block of G with defect group D, one denotes by $\mathrm{Irr}_0(B)$ the set of characters of B with height 0, i.e.,

$$\mathrm{Irr}_0(B) = \{\chi \in \mathrm{Irr}(B) \mid \chi(1)_p = |G : D|_p\} = \{\chi \in \mathrm{Irr}(B) \mid \mathrm{ht}(\chi) = 0\}.$$

When $\nu \in \mathrm{Irr}(N)$ for some $N \lhd G$, let $\mathrm{Irr}_0(B \mid \nu) := \mathrm{Irr}_0(B) \cap \mathrm{Irr}(G \mid \nu)$.

A blockwise analogue of the McKay conjecture was given by Alperin and considers height 0 characters of blocks instead of characters with p'-degree, see [A76].

Conjecture 4.7 (Relative Alperin-McKay Conjecture)**.** *Let G be a finite group, $N \lhd G$, $B \in \mathrm{Bl}(G)$, D a defect group of B and $\nu \in \mathrm{Irr}_D(N)$. Then*

$$(\mathbf{AM}) \qquad |\mathrm{Irr}_0(B \mid \nu)| = |\mathrm{Irr}_0(B' \mid \nu)|,$$

where $B' \in \mathrm{Bl}(\mathrm{N}_G(D)N)$ with $(B')^G = B$.

Proposition 4.8. *Let $N \lhd G$. Then*

(a) $\mathrm{ht}(\chi) \geq \mathrm{ht}(\theta)$ *for every* $\theta \in \mathrm{Irr}(N)$ *and* $\chi \in \mathrm{Irr}(G \mid \theta)$.

(b) Let $(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$ with $\mathrm{ht}(\theta) = \mathrm{ht}(\varphi)$. Assume that $M = \mathrm{N}_N(D)$ for some defect group D of $\mathrm{bl}(\theta)$. Then $\mathrm{ht}(\sigma_G(\psi)) = \mathrm{ht}(\psi)$ for every $\psi \in \mathrm{Irr}(G \mid \theta)$, where σ_G is given by $(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$.

Proof. Part (a) is known from [M96, Lem. 2.2]. Part (b) follows from Proposition 3.9 of [NS14]. □

Theorem 4.9 (Generalized Dade-Glauberman-Nagao (DGN) correspondence)**.** *Let K, L be normal subgroups of G with $K \leq L$ and L/K a p-group. Let $b \in \mathrm{Bl}(K)$ be L-invariant and with defect group $D_0 \leq \mathrm{Z}(K)$. Let $B \in \mathrm{Bl}(L \mid b)$ and D a defect group of B with $B' \in \mathrm{Bl}(\mathrm{N}_L(D))$ its Brauer correspondent.*

Then B' covers a unique block $b' \in \mathrm{Bl}(\mathrm{N}_K(D))$ and D_0 is a defect group of b'. Furthermore we get bijections

$$\mathrm{Irr}(b) \longrightarrow \mathrm{Irr}(D_0) \longrightarrow \mathrm{Irr}(b')$$

hence a bijection $\mathrm{Irr}(b) \longrightarrow \mathrm{Irr}(b')$, $\rho \mapsto \rho^*$.

Proof. By Theorem 5.2 of [NS14] this bijection exists. □

$$\begin{tikzcd}
 & & L \arrow[dd, dash] & B \\
b & K \arrow[ur, dash] \arrow[dd, dash] & & \\
 & & \mathrm{N}_L(D) & B' \\
b' & \mathrm{N}_K(D) \arrow[ur, dash] & &
\end{tikzcd}$$

Remark 4.10. If $p \nmid |K|$, the statement amounts to the classical result due to Glauberman (see Theorem 3.4 above). When $D_0 = 1$ this is the DGN-correspondence (see [NT, §5.12]). The map * is equivariant for automorphisms since it is natural.

Corollary 4.11 ([NS14, Cor. 5.14])**.** *There exists some $\mathrm{N}_{G_B}(D)$-equivariant bijection*

$$\Lambda : \mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(B')$$

such that $(G_\theta, L, \theta) \geq_{\mathrm{b}} (\mathrm{N}_G(D)_\theta, \mathrm{N}_L(D), \Lambda(\theta))$ for any $\theta \in \mathrm{Irr}_0(B)$.

Definition 4.12 (Inductive Alperin-McKay conditions, AM-good)**.** Let S be a non-abelian simple group with universal covering group X and $B \in \mathrm{Bl}(X)$ with defect group Q. Assume that Q is non-central and that for $\Gamma := \mathrm{Aut}(X)_{Q,B}$ there exist

(i) some Γ-stable subgroup M with $\mathrm{N}_X(Q) \leq M \lneq X$,

(ii) some Γ-equivariant bijection $\Lambda : \mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(B')$ where $B' \in \mathrm{Bl}(M)$ is such that $(B')^X = B$ and

(iii) $\Lambda(\mathrm{Irr}_0(B \mid \nu)) \subseteq \mathrm{Irr}_0(B' \mid \nu)$ for every $\nu \in \mathrm{Irr}(\mathrm{Z}(X))$ and

$$(X/Z \rtimes \Gamma_\chi, X/Z, \overline{\chi}) \geq_{\mathrm{b}} (M/Z \rtimes \Gamma_\chi, M/Z, \overline{\Lambda(\chi)})$$

for every $\chi \in \mathrm{Irr}_0(B)$ and $Z = \ker(\chi_{\mathrm{Z}(X)})$, where $\overline{\chi}$ and $\overline{\Lambda(\chi)}$ lift to χ and $\Lambda(\chi)$, respectively.

Then B is said to be *AM-good for p*.

If all p-blocks of X with non-central defect are *AM-good*, X and S are said to be *AM-good*.

The following reduction theorem is the main result of [S13a].

Theorem 4.13. *Let G be a finite group and $B \in \mathrm{Bl}(G)$. Assume that all non-abelian simple groups involved in G are AM-good. Then the relative AM-conjecture 4.7 holds for B.*

Sketch of proof. One uses induction on $|G/\mathrm{Z}(G)|$ and then on $|G|$.

We consider the relative AM-Conjecture. Let D be a defect group of B. Arguing as in Sect. 3.B (see also [M04]), we see that we can assume the group theoretic conditions:

- $\mathrm{N}_G(D)N = G$
- B covers a unique p-block b of N.

If $\mathrm{O}_p(G) \not\leq \mathrm{Z}(G)$, the first property implies then the statement.

If there is some $N \lhd G$ with $N \gneq \mathrm{Z}(G)$ such that B covers a p-block b of N with central defect, one can apply the generalized DGN-correspondence from Theorem 4.9. Thanks to Corollary 4.11 we get a bijection $^* : \mathrm{Irr}_0(b) \longrightarrow \mathrm{Irr}_0(b')$ for some $b' \in \mathrm{Bl}(\mathrm{N}_N(D))$ with $b'^N = b$. By the properties of the relation $\geq_{\mathrm{b}}$ this implies $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(B')|$ for the Brauer correspondent $B' \in \mathrm{Bl}(\mathrm{N}_G(D))$ of B, see Proposition 4.8(b) and Remark 4.3(b-c).

Otherwise there exists a group $L \lhd G$ with $\mathrm{Z}(G) \leq L$ and $L/\mathrm{Z}(G) \cong S^m$ (S non-abelian simple, $m \geq 1$) such that the block C of L covered by B has a non-central defect. One applies in this situation the inductive AM-conditions for S: Via some epimorphism $\epsilon : X^m \twoheadrightarrow L_0 = [L, L]$ the block C relates to some $B_1 \times \cdots \times B_m$ with $B_i \in \mathrm{Bl}(X)$.

$$\begin{array}{ccccc} & & & & G \\ & & & & | \\ & & & & L \\ & & & & | \\ X & \hookrightarrow & X^m & \twoheadrightarrow & L_0 = [L, L] \end{array}$$

Like in the proof of Theorem 3.15 the inductive AM-condition on S allows to control $\mathrm{Irr}_0(C)$. $\square$

Corollary 4.14 (Navarro-Späth [NS14])**.** *Let G be a finite group. Assume that all non-abelian simple groups involved in G satisfy the inductive Alperin-McKay condition. Let $B \in \mathrm{Bl}(G)$ with defect group D and Brauer correspondent $B' \in \mathrm{Bl}(\mathrm{N}_G(D))$ and let $\Gamma := \mathrm{Aut}(G)_{B,D}$. Then there exists some Γ-equivariant bijection $\Lambda : \mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(B')$ such that*

$$(G \rtimes \Gamma_\chi, G, \chi) \geq_{\mathrm{b}} (\mathrm{N}_G(D) \rtimes \Gamma_\chi, \mathrm{N}_G(D), \Lambda(\chi)) \text{ for any } \chi \in \mathrm{Irr}_0(B).$$

This means that the inductive AM-condition implies a "similar" condition on the group. This is a strong version of the Alperin-McKay conjecture for quasi-simple groups and it implies as well a strong version of the Alperin-McKay conjecture for all finite groups.

Remark 4.15. Depending on the nature of the defect groups simplifications in the checking of AM-goodness of a given block are possible. For example, blocks with cyclic defect groups are AM-good (see [KS16]).

In order to verify the condition (iii) from Definition 4.13 it has turned out to be useful to know the following group introduced by Dade.

Definition 4.16 (Dade's ramification group [D73])**.** Let $N \lhd G$, $b \in \mathrm{Bl}(N)$ a p-block assumed to be G-invariant. Then $G[b]$ is the group of elements of G that induce - by conjugation - an inner automorphism of the algebra b.

From this definition it follows that $G[b]$ is a normal subgroup of G containing N. For every group J with $N \leq J \leq G$ we denote by $\mathrm{Bl}(J \mid b)$ the blocks of J covering b.

Proposition 4.17 (Murai)**.** *Let $N \lhd G$ and $b \in \mathrm{Bl}(N)$ a G-invariant p-block. Then:*

(a) $G[b] = \{x \in G \mid \lambda_{\widehat{B}}(\mathrm{cc}_{\langle N,x\rangle}(y)^+) \neq 0$ for some $y \in Nx\}$ where $\widehat{B} \in \mathrm{Bl}(\langle N, x\rangle)$ covers b,

(b) there is a bijection

$$\mathrm{Bl}(G[b] \mid b) \longrightarrow \mathrm{Bl}(G \mid b)$$

defined by $B' \mapsto (B')^G$.

Proof. The formula in part (a) is proven in [M13, 3.2]. Murai gives also there an elegant proof of the statement in (b), that was already known from [D73]. □

This group has the following surprising property.

Proposition 4.18. *Let $N \lhd G$, $b \in \mathrm{Bl}(N)$ with defect group D, $B' \in \mathrm{Bl}(\mathrm{N}_N(D))$ the Brauer correspondent and $\nu \in \mathrm{Irr}(\mathrm{Z}(N))$, $\theta \in \mathrm{Irr}(b \mid \nu)$, $\varphi \in \mathrm{Irr}(b' \mid \nu)$. Assume $G[b] = G$ and $p \nmid |G/N|$. Then $(G, N, \theta) \geq_{\mathrm{b}} (\mathrm{N}_G(D), \mathrm{N}_N(D), \varphi)$.*

Sketch of proof. This follows from [KS15, Thm. C(a)]: By Theorem 1.12 one can replace G by a central extension $\widehat{G}$ of G such that θ, seen as character of some normal subgroup of $\widehat{G}$ extends to some character $\widehat{\theta} \in \mathrm{Irr}(\widehat{G})$. Now using the characterization of Proposition 4.17 of $G[b]$ one can prove that there exists an extension $\widehat{\varphi}$ of φ to $\mathrm{N}_G(D)$. □

4.C. Alperin weight conjecture, Navarro-Tiep reduction theorem. In contrast to what has been seen until now, the Alperin weight conjecture is about modular representations. One denotes by $\mathrm{IBr}(G)$ the set of p-modular Brauer characters, i.e. simple kG-modules up to isomorphism. This set splits along p-blocks $\mathrm{IBr}(G) = \cup_{B \in \mathrm{Bl}(G)} \mathrm{IBr}(B)$.

Most of what has been said above about complex characters – character triples, the associated projective k-representations, the relation $\geq_{\mathrm{b}}$ – is in fact easily transferred to the modular context thanks to Clifford theory (Theorems 1.1, 1.2, and 1.15 above) being valid in positive characteristic.

Definition 4.19. Let (G, N, θ) and (H, M, φ) be modular character triples such that $G = NH$, $N \cap H = M$ and $\mathrm{C}_G(N) \leq H$. Then we write

$$(G, N, \theta) \geq_{\mathrm{c}} (H, M, \varphi)$$

if there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ associated with θ and φ such that their factor sets α and α' satisfy $\alpha_{H \times H} = \alpha'$, and the maps σ_J defined by $(\mathcal{P}, \mathcal{P}')$ satisfy

$$\mathrm{IBr}(\psi_{\mathrm{C}_G(N)}) = \mathrm{IBr}(\sigma_J(\psi)_{\mathrm{C}_G(N)}) \text{ for every } \psi \in \mathrm{IBr}(J \mid \theta)$$

and every group J with $N \leq J \leq G$. We say that

$$(G, N, \theta) \geq_{\mathrm{b}} (H, M, \varphi)$$

if in addition

- a defect group D of $\mathrm{bl}(\varphi)$ satisfies $\mathrm{C}_G(D) \leq H$.
- and the maps σ_J satisfy $\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi)$ for every $\psi \in \mathrm{IBr}(J \mid \theta)$ and $N \leq J \leq G$.

Alperin's conjecture singles out the set $\mathrm{Alp}(B)$ of *p-weights* of a given p-block of G. A p-weight is a G-conjugacy class of pairs (Q,π) where Q is a p-subgroup of G, $\pi \in \mathrm{IBr}(\mathrm{N}_G(Q))$ is such that $\mathrm{bl}(\pi)$ has defect group Q and $\mathrm{bl}(\pi)^G = B$.

Alperin's weight Conjecture. *For any p-block B of any finite group, one has*

$$(\mathbf{AWC}) \qquad |\,\mathrm{IBr}(B)| = |\,\mathrm{Alp}(B)|.$$

Remark 4.20. When the pair (Q,π) determines a p-weight, then π corresponds to a block of defect zero of $\mathrm{N}_G(Q)/Q$.

Definition 4.21 (Inductive Alperin weight condition, AWC-good)**.** Let S be a simple non-abelian group with X its universal covering group and $B \in \mathrm{Bl}(X)$ with non-central defect group. Then B is said to be *AWC-good* if for $\Gamma := \mathrm{Aut}(X)_B$

(i) there exists a Γ-equivariant bijection $\Omega : \mathrm{IBr}(B) \longrightarrow \mathrm{Alp}(B)$ such that

(ii) for every $\chi \in \mathrm{IBr}(B)$ and $\Omega(\chi)$ the X-conjugacy class of some pair (Q,π), one has

$$(X \rtimes \Gamma_{Q,\chi}, X, \chi) \geq_{\mathrm{b}} (\mathrm{N}_X(Q) \rtimes \Gamma_{Q,\chi}, \mathrm{N}_X(Q), \pi).$$

A finite simple group is said to be *AWC-good* if all the blocks with non-central defect of its universal covering group are AWC-good.

The reduction theorem is then easily phrased.

Theorem 4.22 (Navarro-Tiep, Späth)**.** *Let G be a finite group such that every non-abelian simple group involved in G is AWC-good. Then the Alperin weight conjecture holds for all blocks of G.*

Proof. See [NT11] and [S13b]. □

4.D. Navarro-Tiep theorem: the case of groups of Lie type in the defining characteristic.. Before discussing why groups of Lie type satisfy the (non-blockwise) inductive Alperin weight condition we give a Lemma on character triples. Because of the later applications we change slightly the notation.

Lemma 4.23. *Let (J,N,θ) and (H,M,φ) be two modular character triples such that $J = NH$, $M = N \cap H$ and $\mathrm{C}_J(N) \leq H$. Assume that θ is the Brauer character of the simple module V, and that the restriction V_M has a unique submodule with character φ. Then $(J,N,\theta) \geq_{\mathrm{c}} (H,M,\varphi)$.*

Proof. First assume that θ extends to some $\widetilde{\theta} \in \mathrm{IBr}(J)$. Accordingly V can be seen as a J-module $\widetilde{V}$. In $\widetilde{V}_H$ there exists some simple submodule $\widetilde{V}'$ such that $\widetilde{V}'_M$ has V' as submodule (see [Na1, 8.5]). Since φ is H-invariant, we get $\widetilde{V}'_M = V'$. So we get a character $\widetilde{\varphi} \in \mathrm{IBr}(H)$ extending φ. By those definitions $\widetilde{\varphi}_{\mathrm{C}_J(N)}$ and $\widetilde{\theta}_{\mathrm{C}_J(N)}$ are multiples of the same character. Then Lemma 2.15 gives our claim.

Now note that by an analogue of Theorem 1.12 for Brauer characters there exists a central extension $\pi : \widehat{J} \to J$ of J by a cyclic group Z such that θ seen as character θ_0 of a normal subgroup N_0 of $\widehat{J}$ extends to $\widehat{J}$. Here we identify N with a normal subgroup N_0 of $\widehat{J}$, and we call M_0 the subgroup of N corresponding to M. By the above considerations $(\widehat{J}, N_0, \theta_0) \geq_{\mathrm{c}} (\widehat{H}, M_0, \varphi_0)$. By definition $J = \widehat{J}/Z$ and $H = \widehat{H}/Z$. Accordingly the arguments of the proof of Proposition 2.14 imply the statement. □

Remark 4.24. The non-blockwise Alperin weight conjecture considers $\mathrm{Alp}(G) = \cup_{B \in \mathrm{Bl}(G)} \mathrm{Alp}(B)$ and claims

$$(\mathbf{AWC}) \qquad |\,\mathrm{IBr}(G)| = |\,\mathrm{Alp}(G)|.$$

Navarro-Tiep's paper [NT11] gives a reduction theorem for that conjecture. It can be expressed in our terms by replacing $\geq_{\mathrm{b}}$ with $\geq_{\mathrm{c}}$ in Definition 4.21 and Theorem 4.22 above.

Let S be a simple group of Lie type over a finite field of characteristic p. Then, S can be shown to satisfy the hypothesis of this reduction theorem for AWC with regard to that prime p. First its universal covering X is such that $X/\mathrm{Z}(X)_p$ is a group $G = \mathbf{G}^F$ for $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism of a reductive group (in fact $\mathrm{Z}(X)_p = 1$ except for a finite number of exceptions, see [GLS, 6.1.3]). Then the bijection $V \mapsto (Q, \pi)$ given in [C17, §3.C] has the required properties to apply Lemma 4.23 (with $N = G$, $M = \mathrm{N}_G(Q)$) since π is there the Brauer character of the module of fixed points under Q in the simple G-module V.

Dealing with blocks and showing that S actually satisfies the stronger condition of AWC-goodness (Definition 4.21) requires more detailed considerations. A key argument is then that any Sylow p-subgroup of G has a trivial centralizer in $\mathrm{Aut}(G)$, see the complete proof in [S13b, p. 217].

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School of Mathematics and Natural Sciences, University of Wuppertal, Gaußstr. 20, 42119 Wuppertal, Germany
E-mail: bspaeth@uni-wuppertal.de